Prof. G. Raithel

Final exam

Problem 1 **20 Points**

a): In view of part b) of the problem, we seek a solution that involves sines and/or exponentials in the x-and y-directions rather than Bessel functions. Thus, we'll choose cartesian coordinates. Without the boundary at $z = 0$, the free-space eigenfunctions are, as explained in Jackson Eq. 3.162,

$$const. \times \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}) =: const. \times \exp(\mathrm{i}k_z z) \exp(\mathrm{i}\mathbf{k}_{||} \cdot \mathbf{x})$$

where for later convenience we split $\mathbf{k}$ into components transverse (k_z) and parallel ($\mathbf{k}_{||} = \mathbf{k} - \hat{\mathbf{z}}k_z$). These can just as well written in the form

$$const. \times \sin(k_z z) \exp(\mathrm{i}\mathbf{k}_{||} \cdot \mathbf{x}) \quad \text{and} \quad const. \times \cos(k_z z) \exp(\mathrm{i}\mathbf{k}_{||} \cdot \mathbf{x})$$

with $k_z > 0$. For the given geometry, the eigenfunctions with a sine in the z-direction form a complete set. To determine the normalization constant, we write

$$\begin{aligned}
& const^2 \int\int\int_{z>0} \sin(k_z z) \exp(\mathrm{i}\mathbf{k}_{||} \cdot \mathbf{x}) \sin(k'_z z) \exp(-\mathrm{i}\mathbf{k}'_{||} \cdot \mathbf{x}) d^3x \\
= \; & const^2 (2\pi)^2 \delta^2(\mathbf{k}_{||} - \mathbf{k}'_{||}) \int_{z>0} \sin(k_z z) \sin(k'_z z) dz \\
= \; & const^2 2\pi^2 \delta^2(\mathbf{k}_{||} - \mathbf{k}'_{||}) \left(\frac{1}{2\mathrm{i}}\right)^2 \int_z [\exp(\mathrm{i}k_z z) - \exp(-\mathrm{i}k_z z)] [\exp(\mathrm{i}k'_z z) - \exp(-\mathrm{i}k'_z z)] \, dz \\
= \; & const^2 \frac{\pi^2}{2} \delta^2(\mathbf{k}_{||} - \mathbf{k}'_{||}) \int_z [\exp(\mathrm{i}(k_z - k'_z)z) + \exp(-\mathrm{i}(k_z - k'_z)z)] \, dz \\
= \; & const^2 \frac{\pi^2}{2} \delta^2(\mathbf{k}_{||} - \mathbf{k}'_{||}) 4\pi\delta(k_z - k'_z) \\
= \; & const^2 2\pi^3 \delta^3(\mathbf{k} - \mathbf{k}') = \delta^3(\mathbf{k} - \mathbf{k}')
\end{aligned}$$

and thus $const = 1/\sqrt{2\pi^3}$. Thus,

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{2\pi^3}} \sin(k_z z) \exp(\mathrm{i}\mathbf{k}_{||} \cdot \mathbf{x})$$

with $k = \sqrt{\left|k_{||}\right|^2 + k_z^2}$ and $k_x \in \{-\infty, +\infty\}$, $k_y \in \{-\infty, +\infty\}$, and $k_z \in \{0, +\infty\}$. The eigenvalue equation,

$$(\Delta + \lambda)\psi_{\mathbf{k}} = (-k^2 + \lambda) = 0$$

then shows $\lambda = k^2$. From that it follows that

$$G(\mathbf{x},\mathbf{x}') = \frac{2}{\pi^2}\int_{k_x=-\infty}^{k_x=\infty}\int_{k_x=-\infty}^{k_y=\infty}\int_{k_z=0}^{k_z=\infty} \frac{\sin(k_z z)\sin(k_z z')}{\left|k_{||}\right|^2 + k_z^2} \exp\left[\mathrm{i}\mathbf{k}_{||}\cdot(\mathbf{x}-\mathbf{x}')\right] d^3k$$

Note the different lower integration limits in k_x, k_y and k_z.

Note. Though cylindrical coordinates would have been a less fortunate choice to continue with in part b), a correct solution carries full score on part a). In partial analogy with one of the problems in homework set 9, it would work like that:

Find un-normalized eigenfunctions: Write $\psi(\rho,z,\phi) = R(\rho)\Phi(\phi)Z(z)$. Then,

$$\begin{aligned}(\Delta+\lambda)R\Phi Z &= 0\\ \frac{1}{R}\left[d_\rho^2 + \frac{1}{\rho}d_\rho\right]R + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} + \lambda &= 0\end{aligned}$$

Thus, $Z(z) = \sin(k_z z)$ with $k_z \geq 0$ and

$$\frac{\rho^2}{R}\left[d_\rho^2 + \frac{1}{\rho}d_\rho\right]R + \frac{\Phi''}{\Phi} + (\lambda - k_z^2)\rho^2 = 0$$

Thus, $\Phi(\phi) = \exp(\mathrm{i}m\phi)$ with $m = 0, \pm1, \pm2...$, and

$$\left[d_\rho^2 + \frac{1}{\rho}d_\rho + \lambda - k_z^2 - \frac{m^2}{\rho^2}\right]R = 0$$

Finally, set $\lambda - k_z^2 = k_{||}^2$ with $k_{||} > 0$ to find the unnormalized eigenfunction (see Eqns. 3.75ff in Jackson)

$$\psi_{m,k_z,k_{||}}(\rho,z,\phi) \;=\; \exp(\mathrm{i}m\phi)\,\sin(k_z z)\,J_m(k_{||}\rho)$$

with eigenvalue $\lambda = k_z^2 + k_{||}^2$. To normalize, write

$$\begin{aligned}\int \psi^*_{m',k_z',k_{||}'}(\rho,z,\phi)\psi_{m,k_z,k_{||}}(\rho,z,\phi)\rho d\rho d\phi dz &= 2\pi\delta_{m,m'}\frac{1}{k_{||}}\delta(k_{||}-k_{||}')\int_{z=0}^{\infty}\sin(k_z z)\sin(k_z' z)dz\\ &= 2\pi\delta_{m,m'}\frac{1}{k_{||}}\delta(k_{||}-k_{||}')\frac{\pi}{4}\delta(k_z - k_z')\\ &= \frac{\pi^2}{2k_{||}}\delta_{m,m'}\delta(k_{||}-k_{||}')\delta(k_z-k_z')\end{aligned}$$

where we have used Eq. 3.108 in Jackson. The normalized eigenfunctions are thus

$$\psi_{m,k_z,k_{||}}(\rho,z,\phi) \;=\; \frac{\sqrt{2k_{||}}}{\pi}\exp(\mathrm{i}m\phi)\,\sin(k_z z)\,J_m(k_{||}\rho)$$

and the Green's function, with Eqn. 3.160 of Jackson,

$$G(\rho, z, \phi, \rho', z', \phi') = \sum_{m=-\infty}^{\infty} \int_{k_z=0}^{\infty} \int_{k_{||}=0}^{\infty} \frac{8k_{||}}{\pi(k_z^2 + k_{||}^2)} \exp(im(\phi-\phi'))\ \sin(k_z z)\sin(k_z z')\ J_{|m|}(k_{||}\rho) J_{|m|}(k_{||}\rho') dk_z dk_{||}$$

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b): I show the calculation with an additional prefactor V_0 in the boundary condition, *i.e.* $V(\mathbf{x}') = V_0 \sin(\mathbf{k}_0 \cdot \mathbf{x}')$ for $z' = 0$. Set $V_0 = 1$ to compare with the exam.

$$\begin{aligned}
\frac{\partial G}{\partial n'} &= -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} \\
&= -\frac{2}{\pi^2} \int_{k_x=-\infty}^{k_x=\infty} \int_{k_x=-\infty}^{k_y=\infty} \int_{k_z=0}^{k_z=\infty} \left.\frac{k_z \sin(k_z z)\cos(k_z z')}{\left|k_{||}\right|^2 + k_z^2} \exp\left[\mathrm{i}\mathbf{k}_{||}\cdot(\mathbf{x}-\mathbf{x}')\right]\right|_{z'=0} d^3k \\
&= -\frac{2}{\pi^2} \int_{k_x=-\infty}^{k_x=\infty} \int_{k_x=-\infty}^{k_y=\infty} \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{\left|k_{||}\right|^2 + k_z^2} \exp\left[\mathrm{i}\mathbf{k}_{||}\cdot(\mathbf{x}-\mathbf{x}')\right] d^3k
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\int V(\mathbf{x}')\frac{\partial G}{\partial n'} da' \\
\Phi(\mathbf{x}) &= \frac{1}{2\pi^3} \int_{k_x=-\infty}^{k_x=\infty} \int_{k_x=-\infty}^{k_y=\infty} \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{\left|k_{||}\right|^2 + k_z^2} V_0 \exp(\mathrm{i}\mathbf{k}_{||}\cdot\mathbf{x}) \int_{x'}\int_{y'} \sin(\mathbf{k}_0\cdot\mathbf{x}')\exp(-\mathrm{i}\mathbf{k}_{||}\cdot\mathbf{x}') d^3k\, dx'\, dy' \\
&= \frac{1}{2\pi^3} \int_{k_x=-\infty}^{k_x=\infty} \int_{k_x=-\infty}^{k_y=\infty} \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{\left|k_{||}\right|^2 + k_z^2} V_0 \exp(\mathrm{i}\mathbf{k}_{||}\cdot\mathbf{x}) \\
&\quad \times \left\{\frac{1}{2\mathrm{i}} \int_{x'}\int_{y'} \left[\exp(\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}') - \exp(-\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}')\right] \exp(-\mathrm{i}\mathbf{k}_{||}\cdot\mathbf{x}') dx'\, dy'\right\} d^3k \\
&= \frac{1}{\mathrm{i}\pi} \int_{k_x=-\infty}^{k_x=\infty} \int_{k_x=-\infty}^{k_y=\infty} \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{\left|k_{||}\right|^2 + k_z^2} V_0 \exp(\mathrm{i}\mathbf{k}_{||}\cdot\mathbf{x}) \left\{\delta(\mathbf{k}_0 - \mathbf{k}_{||}) - \delta(\mathbf{k}_0 + \mathbf{k}_{||})\right\} d^3k \\
&= \frac{1}{\mathrm{i}\pi} \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{\left|k_0\right|^2 + k_z^2} V_0 \left\{\exp(\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}) - \exp(-\mathrm{i}\mathbf{k}_0\cdot\mathbf{x})\right\} dk_z \\
&= \frac{2}{\pi} V_0 \sin(\mathbf{k}_0\cdot\mathbf{x}) \int_{k_z=0}^{k_z=\infty} \frac{k_z \sin(k_z z)}{k_0^2 + k_z^2} dk_z \\
&= \frac{2}{\pi} V_0 \sin(\mathbf{k}_0\cdot\mathbf{x})\ \frac{\pi}{2}\exp(-k_0 z) \\
&= V_0 \sin(\mathbf{k}_0\cdot\mathbf{x})\exp(-k_0 z)
\end{aligned}$$

Note: The integral provided on the exam sheet was missing a factor $\frac{1}{2}$ (I have a good excuse for that; it's wrong in my mathematical formulae handbook).

c): Two checks are necessary: the solution must satisfy the boundary condition, and in the volume of interest it must solve the Laplace equation.

- For $z' = 0$, it is $\Phi(\mathbf{x}') = V_0 \sin(\mathbf{k}_0\cdot\mathbf{x}')\exp(-k_0 z') = V_0 \sin(\mathbf{k}_0\cdot\mathbf{x}') = V(\mathbf{x}')$

- In the volume of interest, the Laplace equation is satisfied, because

$$\nabla^2 \Phi(\mathbf{x}) = V_0 \nabla^2 \left[\sin(\mathbf{k}_0 \cdot \mathbf{x}) \exp(-k_0 z)\right] = V_0 \left[-k_0^2 + k_0^2\right] = 0$$

The result can actually pretty easily guessed that way.

Problem 2 **20 Points**

This problem is a variant on the examples on magnetized and permeable spheres in homogeneous B-fields, presented in the textbook.

a): Since $\mathbf{B} = \mu\mathbf{H}$ and in general $\mathbf{H} \neq 0$, the permeability is $\mu = 0$ and, since $\mu = \mu_0(1+\chi_b)$, the susceptibility is $\chi_b = -1$.

b): There are no free currents; thus the magnetic potential can be used. We use spherical coordinates. The origin is at the center of the sphere, and θ is measured with respect to the direction of the external B-field. Due to azimuthal symmetry, ϕ is unimportant.

Inside: $\Phi_i(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$

Outside: $\Phi_e(r,\theta) = \sum_l (C_l r^{-l-1} + Dr\delta_{l,1}) P_l(\cos\theta)$

The boundary conditions are, due to the absence of free currents, Tangential H-component:

$$\begin{aligned} H_{\theta,i}\big|_{r=a} &= H_{\theta,e}\big|_{r=a} \\ -\frac{1}{a}\left.\frac{\partial\Phi_i}{\partial\theta}\right|_{r=a} &= -\frac{1}{a}\left.\frac{\partial\Phi_i}{\partial\theta}\right|_{r=a} \\ \sum_l A_l a^{l-1} P_l'(\cos\theta) &= \sum_l (C_l a^{-l-2} + D\delta_{l,1}) P_l'(\cos\theta) \end{aligned}$$

where $P_l'(\cos\theta) = \frac{d}{d\theta}P_l'(\cos\theta)$. Normal B-component:

$$\begin{aligned} B_{\theta,i}\big|_{r=a} &= B_{\theta,e}\big|_{r=a} \\ -\mu\left.\frac{\partial\Phi_i}{\partial r}\right|_{r=a} &= -\mu_0\left.\frac{\partial\Phi_i}{\partial r}\right|_{r=a} \\ 0 &= \sum_l \big((-l-1)C_l a^{-l-2} + D\delta_{l,1}\big) P_l(\cos\theta) \end{aligned}$$

Far away: $\hat{\mathbf{z}}B_0 = -\mu_0\nabla\Phi_e(r\to\infty,\theta) = -\mu_0 D$, thus $D = -\frac{B_0}{\mu_0}$.

From the boundary conditions and the independence of the P_l and P_l' it then follows

$$\begin{aligned} A_l a^{l-1} &= (C_l a^{-l-2} - \frac{B_0}{\mu_0}\delta_{l,1}) \quad \text{for} \quad l = 1,2,3... \\ (-l-1)C_l a^{-l-2} &= \frac{B_0}{\mu_0}\delta_{l,1} \quad \text{for} \quad l = 0,1,2,3... \end{aligned}$$

Thus, from the second and then the first line it follows

$$C_l = \begin{cases} 0 & , \quad l \neq 1 \\ \frac{-a^3 B_0}{2\mu_0} & , \quad l = 1 \end{cases} \qquad \text{and} \qquad A_l = \begin{cases} 0 & , \quad l \neq 1 \\ \frac{-3B_0}{2\mu_0} & , \quad l = 1 \end{cases}$$

(For $l = 0$, which is missing from the first boundary condition, we argue that Φ must be continuous across $r = a$; from there it follows $A_0 = 0$)

Result:

$$\begin{aligned}\Phi_i(r,\theta) &= -\frac{3B_0}{2\mu_0} r\cos\theta \quad , \quad r < a \\ \Phi_e(r,\theta) &= \left(-\frac{B_0}{2\mu_0}\frac{a^3}{r^3} - \frac{B_0}{\mu_0}\right) r\cos\theta \quad , \quad r \geq a\end{aligned}$$

The term $\propto r^{-2}$ is that of a magnetic-dipole $\mathbf{m} = \frac{-2\pi a^3 B_0}{\mu_0}\hat{\mathbf{z}}$.

c):

H-field: $\mathbf{H} = -\nabla\Phi$, thus

$$\begin{aligned}\mathbf{H}(r,\theta) &= \frac{B_0}{\mu_0}\frac{3}{2}\hat{\mathbf{z}} \quad , \quad r < a \\ \mathbf{H}(r,\theta) &= \frac{B_0}{\mu_0}\left\{\hat{\mathbf{z}} - \frac{1}{2}\frac{a^3}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \hat{\theta}\sin\theta\right)\right\} \quad , \quad r \geq a\end{aligned}$$

B-field: $\mathbf{B} = \mu\mathbf{H}$, thus

$$\begin{aligned}\mathbf{B}(r,\theta) &= 0 \quad , \quad r < a \\ \mathbf{B}(r,\theta) &= B_0\left\{\hat{\mathbf{z}} - \frac{1}{2}\frac{a^3}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \hat{\theta}\sin\theta\right)\right\} \quad , \quad r \geq a\end{aligned}$$

Magnetization M: $\mathbf{M} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{H}$, thus

$$\begin{aligned}\mathbf{M} &= -\frac{3B_0}{2\mu_0}\hat{\mathbf{z}} \quad , \quad r < a \\ \mathbf{M} &= 0 \quad , \quad r \geq a\end{aligned}$$

The dipole moment of the sphere thus is $\mathbf{m} = -\frac{B_0}{\mu_0}\frac{3}{2}\frac{4\pi}{3}a^3\hat{\mathbf{z}} = -\frac{2\pi B_0 a^3}{\mu_0}\hat{\mathbf{z}}$, in accordance with the previously found value.

Magnetization currents: There is no volume current, because $-\nabla\cdot\mathbf{M} = 0$ everywhere. The surface current density is

$$\mathbf{M}\times\hat{\mathbf{n}} = -\frac{3B_0}{2\mu_0}\hat{\mathbf{z}}\times\hat{\mathbf{n}} = -\frac{3B_0}{2\mu_0}\sin\theta\hat{\phi}$$

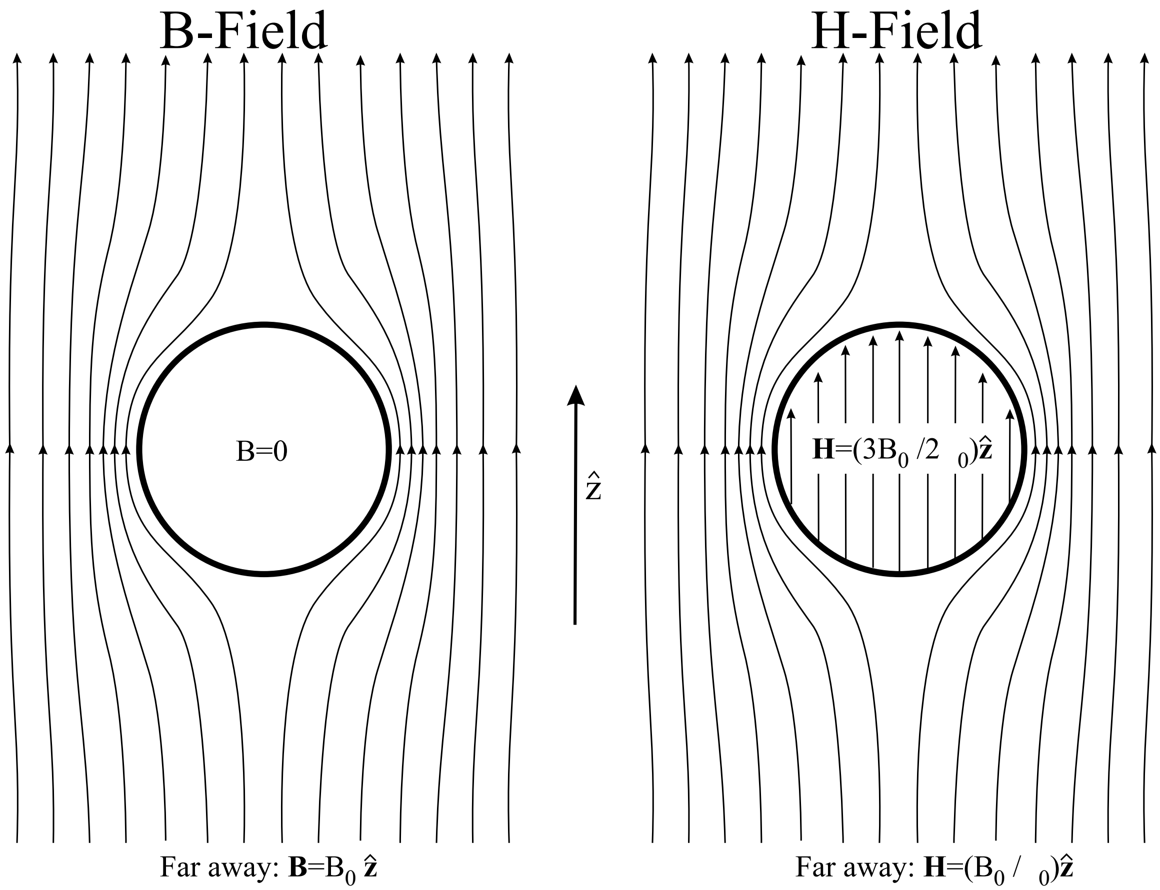

Figure 1: Sketches of the B-(left) and H-fields (right).

Problem 3 **20 Points**

a) Refering to the homework problem Jackson 7.3 and the drawing on the posted solution, we refer to the complex electric field of the upward incident wave at the lower interface as E_0, that of the upward interior wave at the lower interface as E_+, that of the downward interior wave at the lower interface as E_-, that of the transmitted wave at the upper interface as E_t, and that of the reflected wave at the lower interface as E_r. Analogous definitions for the H-fields apply. Directions of positive electric field are out of the plane, directions of positive H-fields of up-going waves are to the right, and directions of positive H-fields of down-going waves are to the left.

For normal incidence, the boundary conditions for the tangential E-component at the lower interface, for the tangential H-component at the lower interface, for the tangential E-component at the upper interface and for the tangential H-component at the upper interface are, in that order,

$$E_+ + E_- = E_0 + E_r \quad (1)$$
$$n(E_+ - E_-) = E_0 - E_r \quad (2)$$
$$E_+ \exp(\mathrm{i}\phi) + E_- \exp(-\mathrm{i}\phi) = E_t \quad (3)$$
$$n(E_+ \exp(\mathrm{i}\phi) - E_- \exp(-\mathrm{i}\phi)) = E_t \quad (4)$$

where $\phi = kd = \frac{\omega n d}{c_0} = \frac{2\pi n d}{\lambda_0}$ with vacuum wavelength λ_0 and speed of light c_0. From the difference and sum of Eqs. 1 and 2 we find

$$E_+ = \frac{1}{2}\left(E_0 \frac{n+1}{n} + E_r \frac{n-1}{n}\right) \quad (5)$$
$$E_- = \frac{1}{2}\left(E_0 \frac{n-1}{n} + E_r \frac{n+1}{n}\right) \quad (6)$$

Inserting into the difference of Eqs. 3 and 4 yields

$$E_+(1-n)\exp(\mathrm{i}\phi) + E_-(n+1)\exp(-\mathrm{i}\phi) = 0$$
$$-E_0(n^2-1)\exp(\mathrm{i}\phi) - E_r(n-1)^2\exp(\mathrm{i}\phi) + E_0(n^2-1)\exp(-\mathrm{i}\phi) + E_r(n+1)^2\exp(-\mathrm{i}\phi) = 0$$
$$\frac{E_r}{E_0}\left[(n+1)^2\exp(-\mathrm{i}\phi) - (n-1)^2\exp(\mathrm{i}\phi)\right] - (n^2-1)\left[\exp(\mathrm{i}\phi) - \exp(-\mathrm{i}\phi)\right] = 0$$
$$\frac{E_r}{E_0}\left[4n\cos\phi - 2\mathrm{i}(n^2+1)\sin\phi\right] - 2\mathrm{i}(n^2-1)\sin\phi = 0$$
$$\frac{E_r}{E_0} = \frac{\mathrm{i}(n^2-1)\sin\phi}{2n\cos\phi - \mathrm{i}(n^2+1)\sin\phi} \quad (7)$$

The intensity reflectivity, $R = \frac{E_r}{E_0}\left(\frac{E_r}{E_0}\right)^*$, thus is

$$R = \frac{(n^2-1)^2\sin^2\phi}{4n^2\cos^2\phi + (n^2+1)^2\sin^2\phi}$$

$$R = \frac{(n^2-1)^2 \sin^2\phi}{4n^2 + (n^2-1)^2 \sin^2\phi} \tag{8}$$

The intensity transmission coefficient T is, due to energy conservation, $T = 1 - R$,

$$T = \frac{4n^2}{4n^2 + (n^2-1)^2 \sin^2\phi} \quad .$$

Note 1: The transmission coefficient can be obtained independently. For instance, "Eq. 3 $+\frac{1}{n}$ Eq. 4 " and "Eq. 3 $-\frac{1}{n}$ Eq. 4 " and subsequent use of Eq. 5 or Eq. 6 yield

$$\frac{E_t}{E_0} = \left(1 + \frac{E_r}{E_0}\left(\frac{n-1}{n+1}\right)\right)\exp(\mathrm{i}\phi) \qquad \text{thus} \qquad \frac{E_r}{E_0} = \left(\frac{E_t}{E_0}\exp(-\mathrm{i}\phi) - 1\right)\left(\frac{n+1}{n-1}\right)$$

$$\frac{E_t}{E_0} = \left(1 + \frac{E_r}{E_0}\left(\frac{n+1}{n-1}\right)\right)\exp(-\mathrm{i}\phi) \qquad \text{thus} \qquad \frac{E_r}{E_0} = \left(\frac{E_t}{E_0}\exp(\mathrm{i}\phi) - 1\right)\left(\frac{n-1}{n+1}\right)$$

Equating the two expressions on the very right yields the above result for T.

Note 2: Defining $c = \exp \mathrm{i}\phi$, the boundary conditions can be written as

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & \frac{1}{n} & 0 \\ c & c^* & 0 & -1 \\ c & -c^* & 0 & -\frac{1}{n} \end{pmatrix} \begin{pmatrix} \frac{E_+}{E_0} \\ \frac{E_-}{E_0} \\ \frac{E_r}{E_0} \\ \frac{E_t}{E_0} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{n} \\ 0 \\ 0 \end{pmatrix}$$

Development using the one of the two rightmost columns, the determinant in easily found to be $D = \frac{4}{n}\cos\phi - 2\mathrm{i}\left(1 + \frac{1}{n^2}\right)\sin\phi$.

Also, the determinant

$$D_3 = \left| \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & \frac{1}{n} & 0 \\ c & c^* & 0 & -1 \\ c & -c^* & 0 & -\frac{1}{n} \end{pmatrix} \right| = 2\mathrm{i}\left(1 - \frac{1}{n^2}\right)\sin\phi$$

and

$$D_4 = \left| \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & \frac{1}{n} & \frac{1}{n} \\ c & c^* & 0 & 0 \\ c & -c^* & 0 & 0 \end{pmatrix} \right| = -\frac{4}{n}$$

Then, $R = \frac{D_3}{D}\left(\frac{D_3}{D}\right)^*$ and $T = \frac{D_4}{D}\left(\frac{D_4}{D}\right)^*$. Results see above.

Note 3: The results can also be obtained by infinite sums. For instance, defining $\phi = 2\pi \frac{2nd}{\lambda_0}$ and using field transmission and reflection coefficients given by Eq. 7.39 for $i = i_0 = 0$ it is

$$E_t = E_0 \left[\frac{4n}{n+1}^2 \sum_{m=0}^{\infty} \left\{ \exp(\mathrm{i}\frac{\phi}{2}) \left(\frac{1-n}{1+n} \right) \right\}^{2m} \right]$$

This is a geometrical sum with result

$$\frac{E_t}{E_0} = \frac{4n}{(n+1)^2 - \exp(\mathrm{i}\frac{\phi}{2})(n-1)^2}$$

which is the same as shown above. A similar sum can be written down for $\frac{E_r}{E_0}$.

Sketch. R alternates between zero and $\left(\frac{n^2-1}{n^2+1}\right)^2$. Zero reflection occurs when $\phi = m\pi$ with integer m, and peak reflection when $\phi = \left(m + \frac{1}{2}\right)\pi$. The reflection minimum at $\phi = 0$ - note that $\phi = 0$ corresponds to vanishing thickness of the layer - can be explained by the phase-jump of π upon reflection at an optically denser medium. The phase jumps occurs even for infinitesimally small layer thickness.

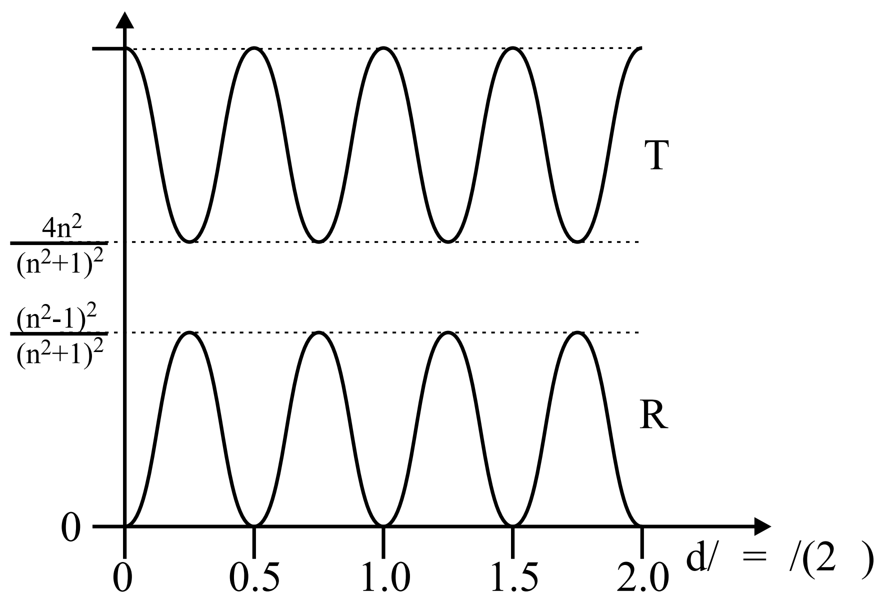

Figure 2: Sketch of the intensity reflection coefficient R and transmission coefficient T.

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b): The refractive indices n_+ and n_- correspond to wavenumbers $k_\pm = \frac{\omega}{c} n_\pm$, respectively. Thus, for waves propagating in the z-direction the complex electric field is of the form

$$\mathbf{E}(z) = E_+ \hat{\epsilon}_+ \exp(\mathrm{i}k_+ z) + E_- \hat{\epsilon}_- \exp(\mathrm{i}k_- z) \quad ,$$

with circular unit vectors $\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} \pm \hat{\mathbf{y}})$, where we have chosen the unit vectors of linear polarization as $\epsilon_1 = \hat{\mathbf{x}}$ and $\epsilon_2 = \hat{\mathbf{y}}$).

The field amplitudes in the circular basis, E_+ and E_-, follow from the condition that the E-field is $\hat{\mathbf{x}}$-polarized at $z = 0$, *i. e.* for $z = 0$

$$\mathbf{E} = E_0\hat{\mathbf{x}} = E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_- = \frac{1}{\sqrt{2}}\left[\hat{\mathbf{x}}(E_+ + E_-) + \mathrm{i}\hat{\mathbf{y}}(E_+ - E_-)\right]$$

Thus, $\frac{1}{\sqrt{2}}(E_+ + E_-) = E_0$ and $E_+ = E_- \Rightarrow E_+ = \frac{1}{\sqrt{2}}E_0$.

The complex electric field vs. z thus is

$$\mathbf{E}(z) = \frac{1}{\sqrt{2}}E_0\left[\hat{\epsilon}_+\exp(\mathrm{i}k_+z) + \hat{\epsilon}_-\exp(\mathrm{i}k_-z)\right] \quad .$$

Defining $k = \frac{1}{2}(k_+ + k_-)$ and $\Delta k = (k_+ - k_-)$, it is seen that $k_\pm = k \pm \frac{\Delta k}{2}$, and $\phi(z) = \frac{\Delta k}{2}z$

$$\begin{aligned}
\mathbf{E}(z) &= \frac{1}{\sqrt{2}}E_0\exp(\mathrm{i}kz)\left[\hat{\epsilon}_+\exp\left(\mathrm{i}\frac{\Delta k}{2}z\right) + \hat{\epsilon}_-\exp\left(-\mathrm{i}\frac{\Delta k}{2}z\right)\right] \\
&= \frac{1}{2}E_0\exp(\mathrm{i}kz)\left[\hat{\mathbf{x}}\left(\exp(\mathrm{i}\phi) + \exp(-\mathrm{i}\phi)\right) + \mathrm{i}\hat{\mathbf{y}}\left(\exp(\mathrm{i}\phi) - \exp(-\mathrm{i}\phi)\right)\right] \\
&= E_0\exp(\mathrm{i}kz)\left[\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}}\right] \\
&= \tilde{E}_0(z)\left[\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}}\right]
\end{aligned}$$

There, the field amplitude $\tilde{E}_0(z)$ has constant magnitude E_0 and includes the average spatial phase shift $\exp(\mathrm{i}kz)$. The polarization is always linear. The polarization angle varies linearly in z and amounts to $\frac{k_- - k_+}{2}z$ vs. the x-axis. Such a polarization rotation is seen, for instance, in the Faraday effect.

Note. In any method used, it first needs to be established that the complex amplitudes E_x and E_y have zero or π phase difference at any location z. Then, it is known that the polarization is always linear. Only then can the magnitudes of E_x and E_y (and the phase difference, zero or π) be used to find the angle of linear polarization.