Physics 505 **Fall 2005**

Practice Final — Solutions

The final will be a 180 minute open book, open notes exam. Do all four problems.

1. A point electric dipole with dipole moment $\vec{p}$ is located at the center of a spherical hole of radius a inside a dielectric medium of infinite extent and dielectric constant ϵ/ϵ_0.

   a) Find the electric field everywhere.

   We first recall that the electric potential for a dipole in isolation is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^2}P_1(\cos\theta)$$

   where we have chosen to orient the dipole along the z axis. This problem is then a standard matching problem, where we may introduce $\Phi_{\rm in}$ for the potential inside the hole ($r<a$) and $\Phi_{\rm out}$ outside. Because of axisymmetry, we write

$$\begin{aligned}\Phi_{\rm in} &= \frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2}+\sum A_l r^l P_l(\cos\theta)\right]\\ \Phi_{\rm out} &= \frac{p}{4\pi\epsilon}\left[\sum\frac{B_l}{r^{l+1}}P_l(\cos\theta)\right]\end{aligned}$$

   where we have already anticipated that the result will be proportional to the dipole moment p (although the constants can be chosen in any arbitrary manner). At $r=a$, we must match both $D^\perp$ and $E^\parallel$. For the spherical hole, the former corresponds to the r direction while the latter the θ direction. Based on orthogonality of the Legendre polynomials, the matching must be done for each l. In this case, we see that there are two equations for two unknowns, A_l and B_l. The $l\neq 1$ terms are homogeneous, yielding only the trivial solution ($A_l=B_l=0$ for $l\neq1$). Thus we only need to focus on $l=1$

$$\begin{aligned}\Phi_{\rm in} &= \frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2}+A_1 r\right]P_1(\cos\theta)\\ \Phi_{\rm out} &= \frac{p}{4\pi\epsilon}\left[\frac{B_1}{r^2}\right]P_1(\cos\theta)\end{aligned}$$

   The matching conditions are then

$$\begin{aligned}D^\perp = -\epsilon\frac{\partial\Phi}{\partial r} &\quad\Rightarrow\quad \left[\frac{2}{a^3}-A_1\right]P_1(\cos\theta) = \frac{2B_1}{a^3}P_1(\cos\theta)\\ E^\parallel = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} &\quad\Rightarrow\quad \frac{1}{\epsilon_0}\left[\frac{1}{a^3}+A_1\right]P_1'(\cos\theta)\sin\theta = \frac{1}{\epsilon}\frac{B_1}{a^3}P_1'(\cos\theta)\sin\theta\end{aligned}$$

This gives a system of linear equations

$$\begin{pmatrix} 1 & 2/a^3 \\ \epsilon/\epsilon_0 & -1/a^3 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 2/a^3 \\ -\epsilon/\epsilon_0 a^3 \end{pmatrix}$$

which may be solved to give

$$A_1 = \frac{2(1-\epsilon_r)}{1+2\epsilon_r}\frac{1}{a^3}, \qquad B_1 = \frac{3\epsilon_r}{1+2\epsilon_r}$$

where $\epsilon_r = \epsilon/\epsilon_0$. Inserting this in to the potential gives

$$\begin{aligned} \Phi_{\text{in}} &= \frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2} + \frac{2(1-\epsilon_r)}{1+2\epsilon_r}\frac{r}{a^3}\right]\cos\theta \\ \Phi_{\text{out}} &= \frac{p}{4\pi\epsilon}\left[\frac{3}{1+2\epsilon_r}\frac{1}{r^2}\right]\cos\theta \end{aligned}$$

which indicates that the field inside the hole is a dipole field superimposed on a uniform field, while the field outside is a pure dipole.

b) What is the bound charge density on the surface of the dielectric?

We may relate the bound charge density to the polarization

$$\sigma_{\text{pol}} = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}$$

where $\hat{n}$ points from region 1 to region 2 and where the polarization is given by

$$\vec{P}_1 = (\epsilon_1 - \epsilon_0)\vec{E}_1, \qquad \vec{P}_2 = (\epsilon_2 - \epsilon_0)\vec{E}_2$$

We take region 1 to be inside the hole and region 2 to be the surrounding dielectric. In this case, we have

$$\hat{n} = \hat{r}, \qquad \epsilon_1 = \epsilon_0, \qquad \epsilon_2 = \epsilon$$

This automatically gives $\vec{P}_1 = 0$, so that

$$\sigma_{\text{pol}} = -\hat{r}\cdot\vec{P}_2 = -(\epsilon-\epsilon_0)\hat{r}\cdot\vec{E}_{\text{out}} = (\epsilon-\epsilon_0)\left.\frac{\partial\Phi_{\text{out}}}{\partial r}\right|_{r=a}$$

Working this out yields

$$\sigma_{\text{pol}} = (\epsilon-\epsilon_0)\frac{p}{4\pi\epsilon_0}\frac{3}{1+2\epsilon_r}\frac{-2}{a^3}\cos\theta = \frac{p}{4\pi a^3}\frac{6(1-\epsilon_r)}{1+2\epsilon_r}\cos\theta$$

The bound charge is distributed in a dipole pattern, which is perhaps not so surprising after all.

2. A long straight wire carrying a current I is located in vacuum parallel to and a distance d away from a semi-infinite slab of permeability μ_r.

   a) Find the magnetic induction everywhere.

   The magnetic induction is a vector quantity. So, to be precise, we take a coordinate system with the interface between dielectrics lying in the x-z plane (at $y = 0$). The y axis is thus normal to the surface of the slab, and we take $y < 0$ to be the material of relative permeability μ_r. The wire is located at $x = 0$ and $y = d$, and carries current in the $+z$ direction. According to Jackson Problem 5.17 (and being careful to switch coordinate systems from 5.17, which had the interface lying in the x-y plane), if we are on the right side, then the original current I gives rise to an image current $I' = [(\mu_r - 1)/(\mu_r + 1)]I$ also flowing in the $+z$ direction, but located at $x = 0$ and $y = -d$. Now, elementary considerations tells us that the magnetic induction due to an infinite straight wire can be written as

$$\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi} = \frac{\mu_0 I}{2\pi R^2}\hat{z} \times \vec{R}$$

   where $\vec{R}$ is the perpendicular distance vector from the wire to the observation point P. More precisely, if $\vec{R}$ is the vector from the real current I to P and $\vec{R}'$ is the vector from the image current I' to P, we have

$$\vec{R} = (x, y - d, 0), \qquad \vec{R}' = (x, y + d, 0)$$

   For $y > 0$, this yields the magnetic induction

$$\begin{aligned}
\vec{B}(x, y > 0) &= \frac{\mu_0 I}{2\pi R^2}\hat{z} \times \vec{R} + \frac{\mu_0 I'}{2\pi R'^2}\hat{z} \times \vec{R}' \\
&= \frac{\mu_0 I}{2\pi}\left[\frac{\hat{z} \times \vec{R}}{R^2} + \frac{\mu_r - 1}{\mu_r + 1}\frac{\hat{z} \times \vec{R}'}{R'^2}\right] \\
&= \frac{\mu_0 I}{2\pi}\left[\frac{(-y + d, x, 0)}{x^2 + (y - d)^2} + \frac{\mu_r - 1}{\mu_r + 1}\frac{(-y - d, x, 0)}{x^2 + (y + d)^2}\right]
\end{aligned} \tag{1}$$

   For $y < 0$, on the other hand, there is no image, and instead the original current I may be replaced by a modified current $I'' = [2\mu_r/(\mu_r + 1)]I$ at the same location $x = 0$, $y = d$. This gives

$$\vec{B}(x, y < 0) = \frac{\mu_0 I''}{2\pi}\frac{\hat{z} \times \vec{R}}{R^2} = \frac{\mu_0 I}{2\pi}\left[\frac{2\mu_r}{\mu_r + 1}\frac{\hat{z} \times \vec{R}}{R^2}\right] = \frac{\mu_0 I}{2\pi}\left[\frac{2\mu_r}{\mu_r + 1}\frac{(-y + d, x, 0)}{x^2 + (y - d)^2}\right]$$

   b) What is the force per unit length on the wire (magnitude and direction)?

   A straightforward method to obtain the force is to realize that it is due to the current I of the actual wire interacting with the magnetic induction $\vec{B}'$ created by its image I'. The magnetic induction from the image is given in (1) as

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left[\frac{\mu_r - 1}{\mu_r + 1}\frac{\hat{z} \times \vec{R}'}{R'^2}\right]$$

Hence the force per unit length is

$$\vec{F}/L = I\hat{z} \times \vec{B} = \frac{\mu_0 I^2}{2\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{\hat{z} \times (\hat{z} \times \vec{R}')}{R'^2}$$

This, of course, is evaluated at the location of the wire, so that $\vec{R}' = (0, 2d, 0) = 2d\hat{y}$. The result is then

$$\vec{F}/L = -\frac{\mu_0 I^2}{2\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{\hat{y}}{2d}$$

The wire is attracted towards its image (the semi-image slab).

3. A rectangular parallel plate capacitor is formed of two flat rectangular conducting sheets of dimensions a and b separated by a distance d. A sinusoidal current $I(t) = I_0 e^{-i\omega t}$ is applied uniformly along the adjacent edges of length b. By orienting the plates so that they lie parallel to the x-y plane with the edge of length b parallel to the y axis, the electric field between the plates can be approximated by $\vec{E} = E_z(x)\hat{z}e^{-i\omega t}$.

   a) Given the above electric field, show that the magnetic induction between the plates can only have a single component $\vec{B} = B_y(x)\hat{y}e^{-i\omega t}$.

   For harmonic fields in a source-free region (between the plates of the capacitor), Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \times \vec{E} - i\omega\vec{B} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} + \frac{i\omega}{c^2}\vec{E} = 0$$

   In particular, we may solve Faraday's law for the magnetic field

$$\vec{B} = -\frac{i}{\omega}\vec{\nabla} \times \vec{E}$$

   For $\vec{E} = E_z(x)\hat{z}$ (we hide the harmonic time dependence), this gives simply

$$\vec{B} = \frac{i}{\omega}E_z'(x)\hat{y} \tag{2}$$

   where the prime denotes an x derivative, d/dx. This clearly shows that the magnetic field only has a y component

$$B_y(x) = \frac{i}{\omega}E_z'(x)$$

   b) Obtain exact expressions for $E_z(x)$ and $B_y(x)$ between the plates. Give your result in terms of Q, the maximum total charge on one plate.

We may obtain a differential equation for $E_z(x)$ by substituting (2) into the Ampère-Maxwell equation

$$0 = \vec{\nabla} \times \vec{B} + \frac{i\omega}{c^2}\vec{E} = \frac{i}{\omega}E_z''(x)\hat{z} + \frac{i\omega}{c^2}E_z(x)\hat{z}$$

This gives

$$E_z''(x) + \frac{\omega^2}{c^2}E_z(x) = 0$$

which is easily solved to yield

$$E_z(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}$$

We now need to make mention of boundary conditions. Although the current feed (which we take at $x = 0$) is not completely symmetric, we assume that the largest amount of charge (in magnitude) must reside closest to the $x = 0$ side of the capacitor, with less charge at the $x = a$ side (since it is furthest away from the feed). Since charge is proportional to the electric field, we want $\vec{E}$ to be largest in magnitude at $x = 0$. This forces the above coefficient B to vanish. To obtain A in terms of the total charge Q, we first compute the surface charge density (on a single plate)

$$\sigma = \epsilon_0 E_z = \epsilon_0 A\cos\frac{\omega x}{c}$$

The total charge is then

$$Q = \int \sigma\, da = \epsilon_0 Ab\int_0^a \cos\frac{\omega x}{c}\, dx = \frac{\epsilon_0 Abc}{\omega}\sin\frac{\omega a}{c}$$

Solving this for A, and writing out the fields results in

$$\begin{aligned} E_z &= \frac{Q}{\epsilon_0}\frac{\omega}{bc\sin\omega a/c}\cos\frac{\omega x}{c} \\ B_y &= -\frac{iQ}{\epsilon_0}\frac{\omega}{bc^2\sin\omega a/c}\sin\frac{\omega x}{c} \end{aligned} \tag{3}$$

c) The resonant frequency of this circuit occurs when the magnitude of the electric field is maximum on the side where the current is fed in and vanishing on the opposite side. What is the resonant frequency?

As indicated, the resonant frequency occurs when the electric field vanishes at $x = a$. Examining (3), this occurs when $\cos\omega a/c = 0$, or when

$$\omega_r = \frac{\pi}{2}\frac{c}{a}$$

Note that higher resonances are possible, but that this is the first one that occurs, and corresponds to $\omega_r = 1/\sqrt{LC}$ in terms of the equivalent circuit elements.

4. A plane polarized electromagnetic wave is incident with angle i on an interface between two dielectrics, with permittivities ϵ and ϵ' (take $\mu = \mu' = \mu_0$). This is the basic system investigated in section 7.3 of Jackson. The transmission and reflection coefficients are defined by

$$T = \left|\frac{E_0'}{E_0}\right|^2, \qquad R = \left|\frac{E_0''}{E_0}\right|^2$$

a) Show that in general $R + T \neq 1$. Consider both cases, $\vec{E}$ perpendicular and parallel to the plane of incidence.

Following the notation of Jackson (and taking unit relative permeability), the transmission and reflection coefficients for the case of $\vec{E}$ perpendicular to the plane of incidence may be written as

$$T = \left(\frac{2n\cos i}{n\cos i + n'\cos r}\right)^2, \qquad R = \left(\frac{n\cos i - n'\cos r}{n\cos i + n'\cos r}\right)^2$$

(assuming all quantities are real). Noting that

$$R = 1 - \frac{4nn'\cos i\cos r}{(n\cos i + n'\cos r)^2}$$

we may write

$$R + T = 1 + \frac{4n\cos i(n\cos i - n'\cos r)}{(n\cos i + n'\cos r)^2} = 1 \pm 2\sqrt{|RT|}$$

The sign is related which one of n or n' is greater, but it hardly matters. The result is that so long as there is reflection and transmission, then generically $R + T \neq 1$.

The case for $\vec{E}$ parallel to the plane of incidence is similar. We have

$$T = \left(\frac{2n\cos i}{n'\cos i + n\cos r}\right)^2, \qquad R = \left(\frac{n'\cos i - n\cos r}{n'\cos i + n\cos r}\right)^2$$

so that

$$R + T = 1 + \frac{4n\cos i(n\cos i - n'\cos r)}{(n'\cos i + n\cos r)^2}$$

This case cannot be put into as elegant a form as above. However, it should be clear that the additional term would not in general vanish. So once again we demonstrate that $R + T \neq 1$.

b) Compute the time averaged Poynting vectors for the incident, reflected and refracted waves. Show that the power per unit area flowing in the direction normal to the interface is conserved. Again consider both polarization cases.

The time averaged Poynting vector is generally written as

$$\vec{S} = \tfrac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\vec{E}\,|^2\hat{k} = \frac{\epsilon}{2n}|\vec{E}\,|^2 c\hat{k}$$

where $\hat{k}$ corresponds to the direction of wave propagation and n is the index of refraction. In this case, the incident, transmitted and reflected Poynting vectors are

$$\begin{aligned}
\vec{S} &= \frac{\epsilon}{2n}|\vec{E}_0|^2 c\hat{k} \\
\vec{S}' &= \frac{\epsilon'}{2n'}|\vec{E}'_0|^2 c\hat{k}' = \frac{\epsilon'}{2n'}T|\vec{E}_0|^2 c\hat{k}' = \frac{\epsilon}{2n}\frac{n'}{n}T|\vec{E}_0|^2 c\hat{k}' \\
\vec{S}'' &= \frac{\epsilon}{2n}|\vec{E}''_0|^2 c\hat{k}'' = \frac{\epsilon}{2n}R|\vec{E}_0|^2 c\hat{k}''
\end{aligned}$$

To proceed, we would like to insert the actual expressions (4) and (5) for T and R into the above. We now consider these two cases in turn.

For $\vec{E}$ perpendicular to the plane of incidence, we use (4) to obtain

$$\begin{aligned}
\vec{S}' &= \frac{\epsilon}{2n}\frac{4nn'\cos^2 i}{(n\cos i + n'\cos r)^2}|\vec{E}_0|^2 c\hat{k}' \\
\vec{S}'' &= \frac{\epsilon}{2n}\frac{(n\cos i - n'\cos r)^2}{(n\cos i + n'\cos r)^2}|\vec{E}_0|^2 c\hat{k}''
\end{aligned}$$

Taking the incident direction to be positive, the power flowing normal to the interface is given by projecting the above onto the unit normal. This gives a factor of either $\cos i$ or $\cos r$. In particular

$$\begin{aligned}
S^{\perp} &= \frac{\epsilon c}{2n}|\vec{E}_0|^2\cos i \\
S'^{\perp} &= \frac{\epsilon c}{2n}\frac{4nn'\cos i\cos r}{(n\cos i + n'\cos r)^2}|\vec{E}_0|^2\cos i \\
S''^{\perp} &= -\frac{\epsilon c}{2n}\frac{(n\cos i - n'\cos r)^2}{(n\cos i + n'\cos r)^2}|\vec{E}_0|^2\cos i
\end{aligned}$$

It is now easy to see that $S^{\perp} + S''^{\perp} = S'^{\perp}$.

Again, the expressions are similar for $\vec{E}$ parallel to the plane of incidence. Using (5), we find

$$\begin{aligned}
\vec{S}' &= \frac{\epsilon}{2n}\frac{4nn'\cos^2 i}{(n'\cos i + n\cos r)^2}|\vec{E}_0|^2 c\hat{k}' \\
\vec{S}'' &= \frac{\epsilon}{2n}\frac{(n'\cos i - n\cos r)^2}{(n'\cos i + n\cos r)^2}|\vec{E}_0|^2 c\hat{k}''
\end{aligned}$$

so that the normal components are

$$
\begin{aligned}
S^{\perp} &= \frac{\epsilon c}{2n}|\vec{E}_0|^2 \cos i \\
S'^{\perp} &= \frac{\epsilon c}{2n}\frac{4nn' \cos i \cos r}{(n' \cos i + n \cos r)^2}|\vec{E}_0|^2 \cos i \\
S''^{\perp} &= -\frac{\epsilon c}{2n}\frac{(n' \cos i - n \cos r)^2}{(n' \cos i + n \cos r)^2}|\vec{E}_0|^2 \cos i
\end{aligned}
$$

The verification of $S^{\perp} + S''^{\perp} = S'^{\perp}$ is almost identical to the perpendicular case above.