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Due date: December 6, 2001

Physics 505: Solutions to Assignment #11

Problem 7.12

(a) The Fourier transforms for charge density $\rho(\vec{r},t)$, current density $\vec{J}(\vec{r},t)$ and the electric field $\vec{E}(\vec{r},t)$ are

$$\rho(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int \rho(\vec{r},t)e^{i\omega t}dt, \qquad \rho(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int \rho(\vec{r},\omega)e^{-i\omega t}d\omega$$

$$\vec{J}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int \vec{J}(\vec{r},t)e^{i\omega t}dt, \qquad \vec{J}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int \vec{J}(\vec{r},\omega)e^{-i\omega t}d\omega$$

$$\vec{E}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int \vec{E}(\vec{r},t)e^{i\omega t}dt, \qquad \vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int \vec{E}(\vec{r},\omega)e^{-i\omega t}d\omega$$

Taking divergence of the Ohm's law:

$$\nabla\cdot\vec{J}(\vec{r},\omega) = \sigma(\omega)\nabla\cdot\vec{E}(\vec{r},\omega),$$

applying the Fourier transformed Gauss's law and the continuity equation:

$$\nabla\cdot\vec{E}(\vec{r},\omega) = \frac{\rho(\vec{r},\omega)}{\epsilon_0}, \qquad \nabla\cdot\vec{J}(\vec{r},\omega) = i\omega\rho(\vec{r},\omega)$$

we have

$$\frac{\sigma(\omega)}{\epsilon_0}\rho(\vec{r},\omega) - i\omega\rho(\vec{r},\omega) = 0 \qquad i.e. \qquad (\sigma(\omega) - i\omega\epsilon_0)\rho(\vec{r},\omega) = 0$$

(b) With

$$\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau} = \frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau}$$

we have

$$\left\{\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} - i\omega\epsilon_0\right\}\rho(\vec{r},\omega) = 0$$

To have non-vanishing charge density, we must have:

$$\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} - i\omega\epsilon_0 = 0 \qquad \Rightarrow \qquad \omega_\pm = \frac{-i\pm\sqrt{4\omega_p^2\tau^2-1}}{2\tau}$$

In the approximation $\omega_p\tau \gg 1$:

$$\omega_\pm = \frac{-i}{2\tau} \pm \omega_p$$

Therefore,

$$\rho(\vec{r},\omega) = \rho_+(\vec{r})\delta(\omega-\omega_+) + \rho_-(\vec{r})\delta(\omega-\omega_-)$$

where ρ_+ and ρ_- are functions determined by initial conditions. The time-dependence of the charge density

$$\rho(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int \rho(\vec{r},\omega)e^{-i\omega t}d\omega = \frac{1}{\sqrt{2\pi}}e^{-t/(2\tau)}\left\{\rho_+(\vec{r})e^{-i\omega_p t} + \rho_-(\vec{r})e^{i\omega_p t}\right\}$$

Therefore, any initial charge distribution will oscillate with the plasma frequency ω_p and decay in amplitude with a decay constant 2τ.

Prob. 7.13

(a) The index of refraction of the ionosphere is

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

The ratios between the amplitudes of reflected and incident wave are given by Eqs. (7.39) and (7.41) for the two polarizations. Note the Eqs. (7.41) and (7.39) have different sign conventions for E_0''.

$$\frac{E_0''}{E_0} = \frac{\cos\theta - n\sin\theta'}{\cos\theta + n\sin\theta'} \quad \text{for } \vec{E} \perp \text{ plane of incidence}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta - \sin\theta'}{n\cos\theta + \sin\theta'} \quad \text{for } \vec{E} \parallel \text{ plane of incidence}$$

In both cases, the amplitude of the ratio is unity when $\sin\theta'$ is imaginary. This corresponds cases that the incidence angle (θ) is greater than the critical angle θ_c:

$$\theta_c = \sin^{-1} n = \sin^{-1}\left\{\frac{\sqrt{\omega^2 - \omega_p^2}}{\omega}\right\}$$

Therefore, the reflection is partial if $\theta < \theta_c$ and is total if $\theta > \theta_c$ for $\omega > \omega_p$.

(b) For simplicity, treat the ionosphere and the earth as flat surfaces and assume that the amateur can only receive distant stations when the wave is totally reflected. In this case,

$$\sin\theta_c = \frac{d}{\sqrt{4h^2 + d^2}} \quad \Rightarrow \quad \frac{\sqrt{\omega^2 - \omega_p^2}}{\omega} = \frac{d}{\sqrt{4h^2 + d^2}} \quad \Rightarrow \quad \omega_p = 2\pi\frac{c}{\lambda}\sqrt{\frac{4h^2}{d^2 + 4h^2}}$$

where $h = 300$ km is the effective height of the F layer, $d = 1000$ km is the distance between the station and the receiver and $\lambda = 21$ m is the wavelength. Plugging in the numbers, we get the plasma frequency

$$\omega_p = 2\pi\frac{3\times 10^8}{21}\sqrt{\frac{4\times(300)^2}{(1000)^2 + 4\times(300)^2}} = 4.6\times 10^7 \text{ Hz}$$

which corresponds to an electron density

$$n = \frac{m\epsilon_0\omega_p^2}{e^2} = 6.6\times 10^{11}/m^3$$

Note the day-night difference is due to the sunlight.

Prob. 7.28

Since the wave has a finite extent in x and y dimensions, the wave is not a plane wave. Assuming the wave is dominated by the transverse polarization, but have a small longitudinal part, the wave can be written as

$$\vec{E}(x,y,z,t) = \{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + F(x,y)\vec{e}_3\}\, e^{i(kz-\omega t)}$$

here $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are unit vectors along $x-$, $y-$ and $z-$axes. The wave must satisfy Maxwell's equation

$$\nabla\cdot\vec{E}(x,y,z,t) = 0 = \left\{\frac{\partial E_0(x,y)}{\partial x} \pm i\frac{\partial E_0(x,y)}{\partial y} + F(x,y)ik\right\} e^{i(kz-\omega t)}$$

Therefore,

$$F(x,y) = \frac{i}{k}\left\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right\}$$

The electric field is then given by

$$\vec{E}(x,y,z,t) = \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k}\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\}\vec{e}_3\right\} e^{i(kz-\omega t)}$$

The magnetic field can be derived from the Maxwell's equation:

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \nabla \times \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k}\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\}\vec{e}_3\right\} e^{i(kz-\omega t)}$$

Since the amplitude modulation is slowly varying, $\partial E_0/\partial x$ and $\partial E_0/\partial y$ are generally small. Neglecting terms of $\partial^2 E_0/\partial x^2$ and $\partial^2 E_0/\partial y^2$, we have

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2)e^{i(kz-\omega t)}\right\}$$

Therefore,

$$\vec{B} = -\frac{i}{\omega}\left\{-\frac{\partial E_2}{\partial z}\vec{e}_1 + \frac{\partial E_1}{\partial z}\vec{e}_2 + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y})\vec{e}_3\right\}$$

$$= -\frac{i}{\omega}\left\{\mp i E_0(x,y)(ik)\vec{e}_1 + E_0(x,y)(ik)\vec{e}_2 + (\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y})\vec{e}_3\right\}$$

$$= -\frac{i}{\omega}(\pm k)\left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + i(\frac{\partial E_0}{\partial x} \pm \frac{\partial E_0}{\partial y})\vec{e}_3\right\}$$

$$= \mp i\frac{k}{\omega}\vec{E} = \mp i\sqrt{\mu\epsilon}\vec{E}$$