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Physics 505: Solutions to Assignment #10

Problem 6.14

(a) The charge on the plate

$$Q(t) = \int_0^t I(t')dt' = \frac{I_0}{\omega}\sin\omega t$$

assuming there is no static charge. In a cylindrical coordinate system, the electric field is along the z and the magnetic field is along the ϕ based on the symmetry of the problem. Let (E_0, E_1) and (B_0, B_1) be the first two non-zero terms in the electric and magnetic field expansions:

$$\vec{E}(\vec{r}) = (E_0 + E_1)\hat{z}; \qquad \vec{B}(\vec{r}) = (B_0 + B_1)\hat{\phi}$$

The first order of the field is given by

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q(t)}{\pi a^2 \epsilon_0} = \frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)$$

From Ampere-Maxwell's law, changing in electric field results in magnetic field:

$$\nabla\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad\Rightarrow\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_0) = \mu_0\epsilon_0\frac{\partial E_0}{\partial t} = \frac{\mu_0 I_0}{\pi a^2}\cos(\omega t) \quad\Rightarrow\quad B_0 = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)$$

Note that there is no static magnetic field. The oscillating magnetic field gives rise to additional electric field:

$$\nabla\times\vec{E}_1 = -\frac{\partial\vec{B}_0}{\partial t} \quad\Rightarrow\quad \frac{\partial E_1}{\partial\rho} = -\frac{\mu_0 I}{2\pi a}\frac{\rho}{a}\omega\sin(\omega t) \quad\Rightarrow\quad E_1 = -\frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega\sin(\omega t)$$

This additional electric field in turn contributes to the magnetic field according to Ampere-Maxwell's law:

$$\nabla\times\vec{B}_1 = \mu_0\epsilon_0\frac{\partial E_1}{\partial t} \quad\Rightarrow\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_1) = -\frac{\mu_0 I_0}{4\pi c^2}\frac{\omega\rho^2}{a^2}\cos(\omega t) \quad\Rightarrow\quad B_1 = -\frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2\cos(\omega t)$$

Combining two contributions together,

$$\vec{E}(\vec{r}) = (E_0 + E_1)\hat{z} = \left\{\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t) - \frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega\sin(\omega t)\right\}\hat{z} = \frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)\left\{1 - \frac{\rho^2}{4c^2}\omega^2\right\}\hat{z}$$

$$\vec{B}(\vec{r}) = (B_0 + B_1)\hat{\phi} = \left\{\frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t) - \frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2\cos(\omega t)\right\}\hat{\phi} = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)\left\{1 - \frac{\rho^2}{8c^2}\omega^2\right\}\hat{\phi}$$

(b) In complex notation, the average energy densities in electric and magnetic fields

$$w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{1}{4}\epsilon_0|E|^2 \quad\Rightarrow\quad w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|B|^2$$

Converting the electric and magnetic fields obtained in (a) into complext notation:

$$\vec{E}(\vec{r}) = Re\left\{i\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\{1 - \frac{\rho^2}{4c^2}\omega^2\}e^{-i\omega t}\right\}\hat{z}$$

$$\vec{B}(\vec{r}) = Re\left\{\frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\{1 - \frac{\rho^2}{8c^2}\omega^2\}e^{-i\omega t}\right\}\hat{\phi}$$

Therefore the total average energies in electric and magnetic fields are:

$$\int w_E d\tau = \frac{1}{4}\epsilon_0(2\pi d)\int_0^a |E|^2\rho d\rho \approx \frac{1}{2}\pi\epsilon_0 d\{\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\}^2\int_0^a(1-\frac{\rho^2}{2c^2}\omega^2)\rho d\rho = \frac{1}{4\pi\epsilon_0}\frac{I_0^2 d}{\omega^2 a^2}\left\{1-\frac{a^2}{4c^2}\omega^2\right\}$$

$$\int w_m d\tau = \frac{1}{4\mu_0}(2\pi d)\int_0^a |B|^2\rho d\rho \approx \frac{\pi d}{2\mu_0}\left\{\frac{\mu_0 I_0}{2\pi a^2}\right\}^2\int_0^a(1-\frac{\rho^2}{8c^2}\omega^2)\rho^3 d\rho = \frac{\mu_0}{32\pi}I_0^2 d\left\{1-\frac{a^2}{6c^2}\omega^2\right\}$$

The total charge on the plate to the second order:

$$Q = \int \sigma da = \int \epsilon_0 E da = 2\pi\epsilon_0\frac{i}{\pi\epsilon_0}\frac{I_0}{\omega a^2}e^{-i\omega t}\int_0^a(1-\frac{\rho^2}{4c^2}\omega^2)\rho d\rho = i\frac{I_0}{\omega}\left\{1-\frac{a^2}{8c^2}\omega^2\right\}e^{-i\omega t}$$

Therefore

$$|I_i|^2 = \omega^2|Q|^2 = I_0^2\left\{1-\frac{a^2}{8c^2}\omega^2\right\}^2 \approx I_0^2\left\{1-\frac{a^2}{4c^2}\omega^2\right\} \quad\Rightarrow\quad I_0^2 \approx |I_i|^2\left\{1+\frac{a^2}{4c^2}\omega^2\right\}$$

Plugging into $\int w_e d\tau$ and $\int w_m d\tau$:

$$\int w_e d\tau = \frac{1}{4\pi\epsilon_0}\frac{d}{\omega^2 a^2}|I_i|^2\left\{1-\frac{a^2}{4c^2}\omega^2\right\}\left\{1+\frac{a^2}{4c^2}\omega^2\right\} \approx \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}$$

$$\int w_m d\tau = \frac{\mu_0 d}{32\pi}|I_i|^2\left\{1+\frac{a^2}{4c^2}\omega^2\right\}\left\{1-\frac{a^2}{6c^2}\omega^2\right\} \approx \frac{\mu_0 d}{4\pi}\frac{|I_i|^2}{8}\left\{1+\frac{a^2}{12c^2}\omega^2\right\}$$

(c) The reactance

$$X = \frac{4\omega}{|I_i|^2}\int(w_m - w_e)d\tau \approx \omega\frac{\mu_0 d}{8\pi} - \frac{1}{\omega}\frac{d}{\pi\epsilon_0 a^2}$$

equivalent to the reactance of an inductor $L = \mu_0 d/8\pi$ and a capacitor $C = \epsilon_0\pi a^2/d$ connected in series. The resonance frequency

$$\omega_{res} = \frac{1}{\sqrt{LC}} = 2\sqrt{2}\frac{c}{a}$$

Problem 7.2

(a) Choose a coordinate system such that the electric field is along the x−axis, the magnetic field along the y−axis and the wave propogates in z−direction. In medium n_1, the incident and reflected waves are described by:

$$\vec{E}^i = E^i e^{i(k_1 z-\omega t)}\hat{x}, \quad \vec{B}^i = \frac{E^i}{v_1}e^{i(k_1 z-\omega t)}\hat{y}$$

$$\vec{E}^r = E^r e^{i(-k_1 z-\omega t)}\hat{x}, \quad \vec{B}^r = -\frac{E^r}{v_1}e^{-i(k_1 z-\omega t)}\hat{y}$$

In medium n_2, there are both forward (denoted as +) and backward (−) propogating waves and are described by:

$$\vec{E}^+ = E^+ e^{i(k_2 z-\omega t)}\hat{x}, \quad \vec{B}^+ = \frac{E^+}{v_2}e^{i(k_2 z-\omega t)}\hat{y}$$

$$\vec{E}^- = E^- e^{i(-k_2 z-\omega t)}\hat{x}, \quad \vec{B}^- = -\frac{E^-}{v_2}e^{-i(k_2 z-\omega t)}\hat{y}$$

In medium n_3, there is only transmitted wave:

$$\vec{E}^t = E^t e^{i(k_3 z - \omega t)}\hat{x}, \qquad \vec{B}^t = \frac{E^t}{v_3} e^{i(k_3 z - \omega t)}\hat{y}$$

where $k_1 = \omega/v_1$, $k_2 = \omega/v_2$, and $k_3 = \omega/v_3$ are wave numbers in the three media. For nonpermeable media ($\mu_1 \approx \mu_2 \approx \mu_3 \approx \mu_0$), $\vec{E}_{||}$ and $\vec{B}_{||}$ are continuous at each interface ($x = 0, d$). At $x = 0$, one has:

$$E^i + E^r = E^+ + E^-; \qquad \frac{E^i - E^r}{v_1} = \frac{E^+ - E^-}{v_2}$$

At $x = d$, one has:

$$E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}; \qquad \frac{E^+ e^{ik_2 d} - E^- e^{-ik_2 d}}{v_2} = \frac{E^t}{v_3} e^{ik_3 d}$$

Let

$$\alpha \equiv \frac{v_1}{v_2} = \frac{n_2}{n_1}; \qquad \beta \equiv \frac{v_2}{v_3} = \frac{n_3}{n_2}$$

The four equations are then

$$E^i + E^r = E^+ + E^-; \qquad E^i - E^r = \alpha(E^+ - E^-)$$

$$E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}; \qquad E^+ e^{ik_2 d} - E^- e^{-ik_2 d} = \beta E^t e^{ik_3 d}$$

Solving for $E^+ e^{ik_2 d}$ and $E^- e^{-ik_2 d}$ from the last two equations:

$$E^+ e^{ik_2 d} = \frac{1}{2}(1+\beta) E^t e^{ik_3 d}, \qquad E^- e^{-ik_2 d} = \frac{1}{2}(1-\beta) E^t e^{ik_3 d}$$

Add the first two equations to eliminate E^r:

$$2E^i = (1+\alpha)E^+ + (1-\alpha)E^- = \frac{1}{2} E^t e^{ik_3 d} \left\{1+\alpha)(1+\beta)e^{-ik_2 d} + (1-\alpha)(1-\beta)e^{ik_2 d}\right\}$$

Solving for E^t in terms of E^i:

$$\frac{E^i}{E^t} = \frac{1}{2} e^{ik_3 d}\{(1+\alpha\beta)\cos(k_2 d) - 2i(\alpha+\beta)\sin(k_2 d)\}$$

Therefore,

$$4|\frac{E^i}{E^t}|^2 = (1+\alpha\beta)^2 \cos^2(k_2 d) + (\alpha+\beta)^2 \sin^2(k_2 d) = (1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)$$

The transmission coefficient T:

$$T = \frac{I^t}{I^i} = \frac{\epsilon_3 v_3 |E^t|^2}{\epsilon_1 v_1 |E^i|^2} = \frac{n_3}{n_1}|\frac{E^t}{E^i}|^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)}$$

$$= \frac{4 n_1 n_2^2 n_3}{n_2^2(n_1+n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}$$

It varies between the two extremum values

$$T_1 = \frac{4 n_1 n_2^2 n_3}{(n_2^2 + n_1 n_3)^2}, \qquad T_2 = \frac{4 n_1 n_3}{(n_1 + n_3)^2}$$

as a function of ω for a fixed d or as a function of d for fixed ω. From the energy conservation, the reflection coefficient R is

$$R = 1 - T = \frac{n_2^2(n_1 - n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}{n_2^2(n_1 + n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}$$

In the special case of $d = 0$, the coefficients reduce to the familiar forms of two media.
(b) For $n_3 = 1$, the reflection coefficient

$$R = \frac{n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}{n_2^2(n_1 + 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}$$

To have zero reflection at $\omega = \omega_0$, the following condition must be satisfied:

$$n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2 d\omega_0/c) = 0$$

Since $n_1 > 1, n_2 > 1$, this is only possible if $n_2 < n_1$. One set of possible solutions is given by

$$\sin^2(n_2 d\omega_0/c) = 1, \qquad \text{and} \qquad n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2) = 0$$

This leads to

$$n_2 = \sqrt{n_1} \qquad \text{and} \qquad d = (\ell + \frac{1}{2})\pi \frac{c}{\sqrt{n_1}\omega_0}$$

where ℓ is a non-zero integer.

Problem 7.3

Note: only need to consider the polarization perpendicular to the plane of incidence and assume $\mu = \mu_0$ in all media.
(a) Assuming the wave incident from left, let E_0 and $E_0^{''}$ be the incident and reflected waves on the left surface, E_+ and E_- be the right and left traveling waves on the left surface in the air gap, θ be the incident angle, and θ' be the refracted angle ($n \sin\theta = \sin\theta'$) in the gap on the left surface, the boundary conditions (parallel components of $\vec{E}$ and $\vec{H}$ continous) on the left surface lead to:

$$E_0 + E_0^{''} = E_+ + E_- \qquad n\cos\theta(E_0 - E_0^{''}) = \cos\theta'(E_+ - E_-)$$

On the right surface, the incident and reflected waves are $E_+ e^{ik\ell}$ and $E_- e^{-ik\ell}$ where $\ell = d/\cos\theta'$ is the path length between the two surfaces and $k = \omega/c$ is the wave number in air. Let E_0' be the transmitted wave, the boundary conditions on the right surface lead to:

$$E_+ e^{ik\ell} + E_- e^{-ik\ell} = E_0' \qquad \cos\theta'(E_+ e^{ik\ell} - E_- e^{-ik\ell}) = n\cos\theta E_0'$$

Defining

$$\alpha = e^{ik\ell} = e^{ikd/\cos\theta'}; \qquad \beta = \frac{n\cos\theta}{\cos\theta'},$$

the four boundary equations are:

$$E_+ + E_- = E_0 + E_0^{''} \qquad E_+ - E_- = \beta(E_0 - E_0^{''})$$

$$\alpha E_+ + \frac{E_-}{\alpha} = E_0^{'} \qquad \alpha E_+ - \frac{E_-}{\alpha} = \beta E_0^{'}$$

Solving these equations for $E_0^{''}$ and $E_0^{'}$:

$$E_0^{'} = \frac{4\alpha\beta}{(1+\beta)^2 - \alpha^2(1-\beta)^2} E_0 = \frac{4\beta e^{i\phi}}{(1+\beta)^2 - e^{2i\phi}(1-\beta)^2} E_0$$

$$E_0^{''} = \frac{(1-\beta^2)(\alpha^2 - 1)}{(1+\beta)^2 - (1-\beta)^2\alpha^2} E_0 = \frac{(1-\beta^2)(e^{2i\phi} - 1)}{(1+\beta)^2 - e^{2i\phi}(1-\beta)^2} E_0$$

where $\phi = kd/\cos\theta' = \omega d/(c\cos\theta')$. The transmission coefficient

$$T = \frac{|E_0^{'}|^2}{|E_0|^2} = \frac{(4\beta)^2}{(1+\beta)^4 + (1-\beta)^4 - 2(1-\beta^2)^2\cos(2\phi)}$$

$$= \frac{(4n\cos\theta\cos\theta')^2}{(n\cos\theta + \cos\theta')^4 + (n\cos\theta - \cos\theta')^4 - 2(n^2\cos^2\theta - \cos^2\theta')^2\cos(2\omega d/(c\cos\theta'))}$$

The reflection coefficient

$$R = \frac{|E_0^{''}|^2}{|E_0|^2} = \frac{2(1-\beta^2)^2(1-\cos\phi)}{(1+\beta)^4 + (1-\beta)^4 - 2(1-\beta^2)^2\cos(2\phi)}$$

$$= \frac{4(n^2\cos^2\theta - \cos^2\theta')^2\sin^2(\omega d/c\cos\theta')}{(n\cos\theta + \cos\theta')^4 + (n\cos\theta - \cos\theta')^4 - 2(n^2\cos^2\theta - \cos^2\theta')^2\cos(2\omega d/(c\cos\theta'))}$$

It is easy to verify that $R + T = 1$.
(b) For $\theta > \theta_c = \sin^{-1}(1/n)$, $\cos\theta' = \sqrt{1-\sin^2\theta'} = \sqrt{1 - n^2\sin^2\theta} = i\sqrt{n^2\sin^2\theta - 1} = i|\cos\theta'|$ is a pure imaginary. The transmission coefficient

$$T = \frac{|4\beta e^{i\phi}|^2}{|(1+\beta)^2 - e^{2i\phi}(1-\beta)^2|^2} = \frac{|4n\cos\theta\cos\theta'|^2 e^{2kd/|\cos\theta'|}}{|(n\cos\theta + \cos\theta')^2 - e^{2kd/|\cos\theta'|}(n\cos\theta - \cos\theta')^2|^2}$$

As $d \to 0$,

$$T \to \frac{|4n\cos\theta\cos\theta'|^2}{|(n\cos\theta + \cos\theta')^2 - (n\cos\theta - \cos\theta')^2|^2} = 1$$

As $d \to \infty$,

$$T \to \frac{|4n\cos\theta\cos\theta'|^2 e^{-2kd/|\cos\theta'|}}{(n^2\cos^2\theta + |\cos\theta'|^2)^2} \to 0$$

as expected.

Problem 7.4
(a) At normal incidence, the reflected wave $E_0^{''}$ is given by

$$\frac{E_0^{''}}{E_0} = \frac{1-n}{1+n}$$

where $n = c/v = c\sqrt{\mu\epsilon}$ is the index of refraction of the medium and E_0 is the incidence wave. For a conductor, $\epsilon \approx i\sigma/\omega$. Therefore,

$$n = c\sqrt{\mu\epsilon} = c\sqrt{i\frac{\mu\sigma}{\omega}} = (1+i)\frac{c}{\omega}\sqrt{\frac{\mu\sigma\omega}{2}} = (1+i)\frac{c}{\omega\delta}$$

where $\delta \equiv \sqrt{2/(\mu\sigma\omega)}$ is the skin depth. Therefore,

$$\frac{E_0^{''}}{E_0} = \frac{1-n}{1+n} = \frac{1-(1+i)c/(\omega\delta)}{1+(1+i)c/(\omega\delta)} = re^{i\phi}$$

where r and ϕ are the amplitude and the phase of the ratio respectively:

$$r = \frac{\sqrt{1+4c^4/(\omega^4\delta^4)}}{1+2c/(\omega\delta)+2c^2/(\omega^2\delta^2)} = \frac{\sqrt{\omega^4\delta^4+4c^4}}{2c^2+2c\omega\delta+\omega^2\delta^2}$$

$$\tan\phi = -\frac{2c/(\omega\delta)}{1 - 2c^2/(\omega^2\delta^2)} = -\frac{2c\omega\delta}{\omega^2\delta^2 - 2c^2}$$

For a perfect conductor, $\sigma \to \infty \;\Rightarrow\; \delta \to 0$, the amplitude and the phase

$$r \to 1 \quad \text{and} \quad \tan\phi \to 0^- \;\; (\phi \to \pi)$$

As expected the reflected wave has a 180° phase change with respect to the incident wave.
(b) The reflection coefficient

$$R = r^2 = \frac{\omega^4\delta^4 + 4c^4}{(2c^2 + 2c\omega\delta + \omega^2\delta^2)^2} \approx \frac{1 + (\omega\delta/c)^4/4}{(1 + \omega\delta/c)^2} \approx 1 - 2\frac{\omega\delta}{c}$$