Physics 505: Solutions to Assignment #10

Problem 6.14
(a) The charge on the plate

\[ Q(t) = \int_0^t I(t') \, dt' = \frac{I_0}{\omega} \sin \omega t \]

assuming there is no static charge. In a cylindrical coordinate system, the electric field is along the \( z \) and the magnetic field is along the \( \phi \) based on the symmetry of the problem. Let \((E_0, E_1)\) and \((B_0, B_1)\) be the first two non-zero terms in the electric and magnetic field expansions:

\[ \vec{E}(\vec{r}) = (E_0 + E_1) \hat{z}; \quad \vec{B}(\vec{r}) = (B_0 + B_1) \hat{\phi} \]

The first order of the field is given by

\[ E_0 = \frac{\sigma}{\varepsilon_0} = \frac{Q(t)}{\pi a^2 \varepsilon_0} = \frac{I_0}{\pi \varepsilon_0 \omega a^2} \sin(\omega t) \]

From Ampere-Maxwell’s law, changing in electric field results in magnetic field:

\[ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_0) = \mu_0 \varepsilon_0 \frac{\partial E_0}{\partial t} = \mu_0 I_0 \frac{\omega}{2\pi a} \omega \sin(\omega t) \Rightarrow B_0 = \frac{\mu_0 I_0 \rho}{2\pi a} \cos(\omega t) \]

Note that there is no static magnetic field. The oscillating magnetic field gives rise to additional electric field:

\[ \nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_0}{\partial t} \Rightarrow \frac{\partial E_1}{\partial \rho} = \frac{\mu_0 I_0 \rho}{2\pi a} \omega \sin(\omega t) \Rightarrow E_1 = -\frac{\mu_0 I_0 \rho^2}{4\pi} \frac{1}{a^2} \omega \sin(\omega t) \]

This additional electric field in turn contributes to the magnetic field according to Ampere-Maxwell’s law:

\[ \nabla \times \vec{B}_1 = \mu_0 \varepsilon_0 \frac{\partial \vec{E}_1}{\partial t} \Rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_1) = \frac{\mu_0 I_0 \omega^2}{4\pi \varepsilon_0} \frac{\rho^2}{a^2} \cos(\omega t) \Rightarrow B_1 = -\frac{\mu_0 I_0 \rho}{16\pi a} \frac{\rho^2}{a^2} \omega^2 \cos(\omega t) \]

Combining two contributions together,

\[ \vec{E}(\vec{r}) = (E_0 + E_1) \hat{z} = \left\{ \frac{I_0}{\pi \varepsilon_0 \omega a^2} \sin(\omega t) - \frac{\mu_0 I_0 \rho^2}{4\pi} \frac{1}{a} \omega \sin(\omega t) \right\} \hat{z} = \frac{I_0}{\pi \varepsilon_0 \omega a^2} \sin(\omega t) \left\{ 1 - \frac{\rho^2}{4c^2 \omega^2} \right\} \hat{z} \]

\[ \vec{B}(\vec{r}) = (B_0 + B_1) \hat{\phi} = \left\{ \frac{\mu_0 I_0 \rho}{2\pi a} \cos(\omega t) - \frac{\mu_0 I_0 \rho^2}{16\pi a^2} \frac{1}{\varepsilon_0} \omega^2 \cos(\omega t) \right\} \hat{\phi} = \frac{\mu_0 I_0 \rho}{2\pi a} \cos(\omega t) \left\{ 1 - \frac{\rho^2}{8c^2 \omega^2} \right\} \hat{\phi} \]

(b) In complex notation, the average energy densities in electric and magnetic fields

\[ w_e = \frac{1}{4} \vec{E} \cdot \vec{E}^* = \frac{1}{4\varepsilon_0} |E|^2 \quad \Rightarrow \quad w_m = \frac{1}{4} \vec{B} \cdot \vec{B}^* = \frac{1}{4\mu_0} |B|^2 \]

Converting the electric and magnetic fields obtained in (a) into complex notation:

\[ \vec{E}(\vec{r}) = \text{Re} \left\{ \frac{I_0}{\pi \varepsilon_0 \omega a^2} \left( 1 - \frac{\rho^2}{4c^2 \omega^2} \right) e^{-i\omega t} \right\} \hat{z} \]

\[ \vec{B}(\vec{r}) = \text{Re} \left\{ \frac{\mu_0 I_0 \rho}{2\pi a} \frac{1}{a} \left( 1 - \frac{\rho^2}{8c^2 \omega^2} \right) e^{-i\omega t} \right\} \hat{\phi} \]
Therefore the total average energies in electric and magnetic fields are:

\[
\int w_E\,d\tau = \frac{1}{4\epsilon_0}(2\pi d) \int_0^a |E|^2 \rho d\rho \approx \frac{1}{2\pi \epsilon_0 a d} \left( \frac{I_0}{\pi \epsilon_0 \omega a^2} \right)^2 \int_0^a (1 - \frac{\rho^2}{2e^2\omega^2}) \rho d\rho = \frac{1}{4\pi \epsilon_0 \omega^2 a^2} \left( 1 \frac{I_0^2 d}{4\pi} \right) \left( 1 - \frac{a^2}{4e^2\omega^2} \right)
\]

\[
\int w_m\,d\tau = \frac{1}{4\mu_0}(2\pi d) \int_0^a |B|^2 \rho d\rho \approx \frac{\pi d}{2\mu_0} \left( \frac{\mu_0 I_0}{2\pi a^2} \right)^2 \int_0^a (1 - \frac{\rho^2}{8e^2\omega^2}) \rho^3 d\rho = \frac{1}{2\pi \mu_0} \left( \frac{I_0^2}{8\pi} \right) \left( 1 - \frac{a^2}{6e^2\omega^2} \right)
\]

The total charge on the plate to the second order:

\[
Q = \int \sigma \,da = \int \epsilon_0 E \,da = 2\pi \epsilon_0 \frac{i}{\pi \epsilon_0} \frac{I_0}{\omega a^2} e^{-i\omega t} \int_0^a (1 - \frac{\rho^2}{4e^2\omega^2}) \rho d\rho = i \frac{I_0}{\omega} \left( 1 \frac{a^2}{4e^2\omega^2} \right) e^{-i\omega t}
\]

Therefore

\[
|I|^2 = \omega^2 |Q|^2 = I_0^2 \left( 1 \frac{a^2}{4e^2\omega^2} \right) \approx I_0^2 \left( 1 \frac{a^2}{4e^2\omega^2} \right) \Rightarrow \frac{I_0^2}{8\pi} \approx |I|^2 \left( 1 + \frac{a^2}{4e^2\omega^2} \right)
\]

Plugging into \(\int w_E\,d\tau\) and \(\int w_m\,d\tau\):

\[
\int w_E\,d\tau = \frac{1}{4\pi \epsilon_0 \omega^2 a^2} |I|^2 \left( 1 \frac{a^2}{4e^2\omega^2} \right) \left( 1 + \frac{a^2}{4e^2\omega^2} \right) \approx \frac{1}{4\pi \epsilon_0 \omega^2 a^2} \left| \frac{I_0^2 d}{4\pi} \right|
\]

\[
\int w_m\,d\tau = \frac{\mu_0 d}{32\pi} |I|^2 \left( 1 \frac{a^2}{4e^2\omega^2} \right) \left( 1 - \frac{a^2}{6e^2\omega^2} \right) \approx \frac{\mu_0 d}{4\pi} \left| I_0^2 \right| \left( 1 \frac{a^2}{12e^2\omega^2} \right)
\]

(c) The reactance

\[
X = \frac{4\omega}{|I|^2} \int (w_m - w_E)\,d\tau \approx \omega \frac{\mu_0 d}{8\pi} - \frac{1}{\omega} \frac{d}{\pi \epsilon_0 a^2}
\]

equivalent to the reactance of an inductor \(L = \mu_0 d/8\pi\) and a capacitor \(C = \epsilon_0 \pi a^2/d\) connected in series. The resonance frequency

\[
\omega_{res} = \frac{1}{\sqrt{LC}} = 2\sqrt{\frac{e}{a}}
\]

**Problem 7.2**

(a) Choose a coordinate system such that the electric field is along the \(x\)–axis, the magnetic field along the \(y\)–axis and the wave propagates in \(z\)–direction. In medium \(n_1\), the incident and reflected waves are described by:

\[
\vec{E}_i = E_i e^{i(kz-\omega t)} \hat{x}, \quad \vec{B}_i = \frac{E_i}{v_1} e^{i(kz-\omega t)} \hat{y}
\]

\[
\vec{E}_r = E_r e^{i(-kz-\omega t)} \hat{x}, \quad \vec{B}_r = -\frac{E_r}{v_1} e^{-i(kz-\omega t)} \hat{y}
\]

In medium \(n_2\), there are both forward (denoted as \(+\)) and backward (\(-\)) propagating waves and are described by:

\[
\vec{E}_+ = E_+ e^{i(kz+\omega t)} \hat{x}, \quad \vec{B}_+ = \frac{E_+}{v_2} e^{i(kz+\omega t)} \hat{y}
\]

\[
\vec{E}_- = E_- e^{i(-kz+\omega t)} \hat{x}, \quad \vec{B}_- = -\frac{E_-}{v_2} e^{-i(kz+\omega t)} \hat{y}
\]
In medium $n_3$, there is only transmitted wave:

$$\vec{E}^t = E^t e^{i(k_z z - \omega t)} \hat{x}, \quad \vec{B}^t = \frac{E^t}{v_3} e^{i(k_z z - \omega t)} \hat{y}$$

where $k_1 = \omega/v_1$, $k_2 = \omega/v_2$, and $k_3 = \omega/v_3$ are wave numbers in the three media. For nonpermeable media ($\mu_1 \approx \mu_2 \approx \mu_3 \approx \mu_0$), $\vec{E}_||$ and $\vec{B}_||$ are continuous at each interface ($x = 0, d$). At $x = 0$, one has:

$$E^i + E^r = E^+ + E^-; \quad \frac{E^i - E^r}{v_1} = \frac{E^+ - E^-}{v_2}$$

At $x = d$, one has:

$$E^+ e^{ikzd} + E^- e^{-ikzd} = E^t e^{ikzd}; \quad \frac{E^+ e^{ikzd} - E^- e^{-ikzd}}{v_2} = \frac{E^t}{v_3} e^{ikzd}$$

Let

$$\alpha = \frac{v_1}{v_2} = \frac{n_2}{n_1}; \quad \beta = \frac{v_2}{v_3} = \frac{n_3}{n_2}$$

The four equations are then

$$E^i + E^r = E^+ + E^-; \quad E^i - E^r = \alpha (E^+ - E^-)$$

$$E^+ e^{ikzd} + E^- e^{-ikzd} = E^t e^{ikzd}; \quad E^+ e^{ikzd} - E^- e^{-ikzd} = \beta E^t e^{ikzd}$$

Solving for $E^+ e^{ikzd}$ and $E^- e^{-ikzd}$ from the last two equations:

$$E^+ e^{ikzd} = \frac{1}{2} (1 + \beta) E^t e^{ikzd}, \quad E^- e^{-ikzd} = \frac{1}{2} (1 - \beta) E^t e^{ikzd}$$

Add the first two equations to eliminate $E^r$:

$$2E^i = (1 + \alpha) E^+ + (1 - \alpha) E^- = \frac{1}{2} E^t e^{ikzd} \left[ 1 + (1 + \beta)(1 - \beta) e^{-ikzd} + (1 - \alpha)(1 - \beta) e^{ikzd} \right]$$

Solving for $E^\parallel$ in terms of $E^\perp$:

$$\frac{E^\parallel}{E^\perp} = \frac{1}{2} e^{ikzd} \left[ (1 + \alpha\beta) \cos(k_2 d) - 2i(\alpha + \beta) \sin(k_2 d) \right]$$

Therefore,

$$4\left| \frac{E^\parallel}{E^\perp} \right|^2 = (1 + \alpha\beta)^2 \cos^2(k_2 d) + (\alpha + \beta)^2 \sin^2(k_2 d) = (1 + \alpha\beta)^2 - (1 - \alpha^2)(1 - \beta^2) \sin^2(k_2 d)$$

The transmission coefficient $T$:

$$T = \frac{I^t}{I^r} = \frac{\varepsilon^3 \mu_0 |E^\parallel|^2}{\varepsilon^1 \mu_1 |E^\perp|^2} = \frac{n_3^3 |E^t|^2}{n_1^3 |E^\perp|^2} = \frac{4\alpha\beta}{(1 + \alpha\beta)^2 - (1 - \alpha^2)(1 - \beta^2) \sin^2(k_2 d)}$$

$$= \frac{4n_1 n_2^2 n_3}{n_2^2 (n_1 + n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_3^2) \sin^2(n_2 d\omega/c)}$$

It varies between the two extremum values:

$$T_1 = \frac{4n_1 n_2^2 n_3}{(n_2^2 + n_1 n_3)^2}, \quad T_2 = \frac{4n_1 n_3}{(n_1 + n_3)^2}$$
as a function of $\omega$ for a fixed $d$ or as a function of $d$ for fixed $\omega$. From the energy conservation, the reflection coefficient $R$ is

$$R = 1 - T = \frac{n_2^2(n_1 - n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2d\omega/c)}{n_2^2(n_1 - n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2d\omega/c)}$$

In the special case of $d = 0$, the coefficients reduce to the familiar forms of two media.

(b) For $n_3 = 1$, the reflection coefficient

$$R = \frac{n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2d\omega/c)}{n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2d\omega/c)}$$

To have zero reflection at $\omega = \omega_0$, the following condition must be satisfied:

$$n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2(n_2d\omega_0/c) = 0$$

Since $n_1 > 1, n_2 > 1$, this is only possible if $n_2 < n_1$. One set of possible solutions is given by

$$\sin^2(n_2d\omega_0/c) = 1, \quad \text{and} \quad n_2^2(n_1 - 1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2) = 0$$

This leads to

$$n_2 = \sqrt{n_1} \quad \text{and} \quad d = (\ell + \frac{1}{2})\pi \frac{c}{\sqrt{n_1}\omega_0}$$

where $\ell$ is a non-zero integer.

**Problem 7.3**

Note: only need to consider the polarization perpendicular to the plane of incidence and assume $\mu = \mu_0$ in all media.

(a) Assuming the wave incident from left, let $E_0$ and $E_0'$ be the incident and reflected waves on the left surface, $E_+$ and $E_-$ be the right and left traveling waves on the left surface in the air gap, $\theta$ be the incident angle, and $\theta'$ be the refracted angle ($n_\sin \theta = \sin \theta'$) in the gap on the left surface, the boundary conditions (parallel components of $E$ and $H$ continuous) on the left surface lead to:

$$E_0 + E_0'' = E_+ + E_- \quad n \cos \theta(E_0 - E_0') = \cos \theta'(E_+ - E_-)$$

On the right surface, the incident and reflected waves are $E_+ e^{ik\ell} + E_- e^{-ik\ell}$ where $\ell = d/\cos \theta'$ is the path length between the two surfaces and $k = \omega/c$ is the wave number in air. Let $E_0'$ be the transmitted wave, the boundary conditions on the right surface lead to:

$$E_+ e^{ik\ell} + E_- e^{-ik\ell} = E_0' \quad \cos \theta'(E_+ e^{ik\ell} - E_- e^{-ik\ell}) = n \cos \theta E_0'$$

Defining

$$\alpha = e^{ik\ell} = e^{ik\ell/\cos \theta}; \quad \beta = \frac{n \cos \theta}{\cos \theta'},$$

the four boundary equations are:

$$E_+ + E_- = E_0 + E_0'' \quad E_+ - E_- = \beta(E_0 - E_0'')$$

$$\alpha E_+ + \frac{E_-}{\alpha} = E_0' \quad \alpha E_+ - \frac{E_-}{\alpha} = \beta E_0'$$

Solving these equations for $E_0''$ and $E_0'$:

$$E_0'' = \frac{4\alpha \beta}{(1 - \beta)^2 - \alpha^2(1 - \beta)^2} E_0 = \frac{4\beta e^{i\phi}}{(1 + \beta)^2 - e^{2i\phi}(1 - \beta)^2} E_0$$

$$E_0' = \frac{(1 - \beta^2)(\alpha^2 - 1)}{(1 + \beta)^2 - (1 - \beta)^2 \alpha^2} E_0 = \frac{(1 - \beta^2)(e^{3i\phi} - 1)}{(1 + \beta)^2 - e^{2i\phi}(1 - \beta)^2} E_0$$
where $\phi = k d / \cos \theta' = \omega d / (c \cos \theta')$. The transmission coefficient

$$ T = \frac{|E''_0|^2}{|E_0|^2} = \frac{(4\beta)^2}{(1 + \beta)^4 + (1 - \beta)^4 - 2(1 - \beta^2)^2 \cos(2\theta')}, $$

$$ = \frac{(4n \cos \theta \cos \theta')^2}{(n \cos \theta + \cos \theta')^4 + (n \cos \theta - \cos \theta')^4 - 2(n^2 \cos^2 \theta - \cos^2 \theta')^2 \cos(2\omega d / (c \cos \theta'))} $$

The reflection coefficient

$$ R = \frac{|E'_0|^2}{|E_0|^2} = \frac{2(1 - \beta^2)(1 - \cos \phi)}{(1 + \beta)^4 + (1 - \beta)^4 - 2(1 - \beta^2)^2 \cos(2\theta')} $$

$$ = \frac{4(n^2 \cos^2 \theta - \cos^2 \theta')^2 \sin^2(\omega d / c \cos \theta')}{(n \cos \theta + \cos \theta')^4 + (n \cos \theta - \cos \theta')^4 - 2(n^2 \cos^2 \theta - \cos^2 \theta')^2 \cos(2\omega d / (c \cos \theta'))} $$

It is easy to verify that $R + T = 1$.

(b) For $\theta > \theta_c = \sin^{-1}(1/n)$, $\cos \theta' = \sqrt{1 - \sin^2 \theta'} = \sqrt{1 - n^2 \sin^2 \theta} = i \sqrt{n \sin^2 \theta - 1} = i |\cos \theta'|$ is a pure imaginary. The transmission coefficient

$$ T = \frac{|4\beta e^{i\theta}|^2}{(1 + \beta)^2 - e^{2i\theta}(1 - \beta)^2} = \frac{|4n \cos \theta \cos \theta' e^{2i\omega d / \cos \theta'}|^2}{|(n \cos \theta + \cos \theta')^2 - e^{2i\omega d / \cos \theta'}(n \cos \theta - \cos \theta')^2|^2} $$

As $d \to 0$,

$$ T \to \frac{|4n \cos \theta \cos \theta'|^2}{|(n \cos \theta + \cos \theta')^2 - (n \cos \theta - \cos \theta')^2|^2} = 1 $$

As $d \to \infty$,

$$ T \to \frac{|4n \cos \theta \cos \theta'|^2 e^{-2i\omega d / \cos \theta'}}{(n^2 \cos^2 \theta - |\cos \theta'|^2)^2} \to 0 $$

as expected.

**Problem 7.4**

(a) At normal incidence, the reflected wave $E''_0$ is given by

$$ \frac{E''_0}{E_0} = \frac{1 - n}{1 + n} $$

where $n = c / v = c / \sqrt{\mu \epsilon}$ is the index of refraction of the medium and $E_0$ is the incidence wave. For a conductor, $\epsilon \approx i\sigma / \omega$. Therefore,

$$ n = c / \sqrt{\mu \epsilon} = c \sqrt{\frac{\mu \sigma}{\omega}} = (1 + i) \frac{c}{\omega} \sqrt{\frac{\mu \sigma}{\omega}} = (1 + i) \frac{c}{\omega \delta} $$

where $\delta \equiv \sqrt{2 / (\mu \sigma \omega)}$ is the skin depth. Therefore,

$$ \frac{E''_0}{E_0} = \frac{1 - n}{1 + n} = \frac{1 - (1 + i) c / (\omega \delta)}{1 + (1 + i) c / (\omega \delta)} = r e^{i \phi} $$

where $r$ and $\phi$ are the amplitude and the phase of the ratio respectively:

$$ r = \frac{\sqrt{1 - 4c^2 / (\omega^2 \delta^2)}}{1 + 2c / (\omega \delta) + 2c^2 / (\omega^2 \delta^2)} = \frac{\sqrt{\omega^4 \delta^4 - 4c^4}}{2c^2 + 2c \omega \delta + \omega^2 \delta^2} $$
\[
\tan \phi = -\frac{2c/(\omega \delta)}{1 - 2c^2/(\omega^2 \delta^2)} = \frac{2c \omega \delta}{\omega^2 \delta^2 - 2c^2}
\]

For a perfect conductor, \( \sigma \to \infty \Rightarrow \delta \to 0 \), the amplitude and the phase

\[ r \to 1 \quad \text{and} \quad \tan \phi \to 0^- \quad (\phi \to \pi) \]

As expected the reflected wave has a 180° phase change with respect to the incident wave.

(b) The reflection coefficient

\[
R = r^2 = \frac{\omega^4 \delta^4 + 4c^4}{(2c^2 + 2c \omega \delta + \omega^2 \delta^2)^2} \approx \frac{1 - (\omega \delta/c)^4}{(1 + \omega \delta/c)^2} \approx 1 - \frac{2 \omega \delta}{c}
\]