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Physics 505: Solutions to Assignment #9

Problem 6.4

(a) Since the sphere is uniformly magnetized with magnetic moment $m = 4\pi R^3 M/3$, the magnetization is therefore M. The magnetic field inside the sphere is then given by Eq. (5.105):

$$\vec{B} = \frac{2}{3}\mu_0 \vec{M}$$

Given the Ohm's law in a moving conductor $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ and the fact that there is no current flowing inside the conductor, the electric field inside the conductor must be:

$$\vec{E} = -\vec{v} \times \vec{B} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = \vec{B} \times (\vec{\omega} \times \vec{r}) = (\vec{B} \cdot \vec{r})\vec{\omega} - (\vec{B} \cdot \vec{\omega})\vec{r}$$

$$= \frac{2}{3}\mu_0 M(r\cos\theta\vec{\omega} - \omega\vec{r}) = \frac{2}{3}\mu_0 M\{z\omega\hat{z} - \omega(x\hat{x} + y\hat{y} + z\hat{z})\} = -\frac{2}{3}\mu_0 M\omega(x\hat{x} + y\hat{y})$$

Here we have chosen Spherical and Cartesian coordinates with their origins at the center of the sphere and their $z-$axis along the magnetization direction. Therefore, the volume charge density is then given by the Coulomb's law:

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = -\frac{4}{3}\epsilon_0\mu_0 M\omega = -\frac{m\omega}{\pi c^2 R^3}$$

The total volume charge

$$Q_\rho = \int \rho d\tau = -\frac{4m\omega}{3c^2}$$

Note that rotating electric charges will result in an additional magnetic field. However, this field is suppressed by a factor of v/c compared with the field from magnetization and therefore ignored. (c) The surface charge distribution is $\phi-$symmetric and therefore can be written as

$$\sigma(\theta) = \sum_\ell \sigma_\ell P_\ell(\cos\theta)$$

Since the conductor is uncharged:

$$\oint \sigma(\theta)da + \int_V \rho d\tau = 0 \quad \Rightarrow \quad \sigma_0 = -\frac{1}{4\pi R^2}\rho\frac{4}{3}\pi R^3 = \frac{m\omega}{3\pi c^2 R^2} = -\frac{Q_\rho}{4\pi R^2}$$

From $\vec{E} = -\vec{v} \times \vec{B} = -(\vec{\omega} \times \vec{r}) \times \vec{B}$, we note that the electric field along the $z-$axis is zero and therefore the scalar potential is constant along the axis. For a point ($r < R$) inside the sphere, the potential due to the volume charge is (can be calculated in a variety of ways such as using Gauss's law to calculate the field first and then integrate):

$$\Phi_\rho(r) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2)$$

On the $z-$axis:

$$\Phi_\rho(z) = \frac{\rho}{6\epsilon_0}(3R^2 - z^2)$$

The potential due to the surface charge for a point on the $z-$axis:

$$\Phi_\sigma(z) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\theta')}{|\vec{r} - \vec{r'}|}da'$$

Note that

$$\frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{R}\frac{1}{\sqrt{1 + (z/R)^2 - 2(z/R)\cos\theta'}} = \sum_\ell \frac{z^\ell}{R^{\ell+1}}P_\ell(\cos\theta')$$

Consequently

$$\Phi_\sigma(z) = \frac{R^2}{2\epsilon_0}\sum_{\ell,\ell'}\frac{z^\ell \sigma_{\ell'}}{R^{\ell+1}}\int P_\ell(\cos\theta')P_{\ell'}(\cos\theta')d\cos\theta' = \frac{R^2}{\epsilon_0}\sum_\ell \frac{\sigma_\ell}{2\ell+1}\frac{z^\ell}{R^{\ell+1}}$$

The combined potential along the $z-$axis:

$$\Phi(z) = \Phi_\rho(z) + \Phi_\sigma(z) = \frac{\rho}{6\epsilon_0}(3R^2 - z^2) + \frac{R^2}{\epsilon_0}\sum_\ell \frac{\sigma_\ell}{2\ell+1}\frac{z^\ell}{R^{\ell+1}}$$

$\Phi(z)$ is independent of z only if

$$\sigma_2 = \frac{5}{6}\rho R = -\frac{5}{6}\frac{m\omega}{\pi c^2 R^2}, \quad \text{and} \quad \sigma_\ell = 0 \ (\ell \neq 0, 2)$$

The surface charge density is then

$$\sigma(\theta) = \sigma_0 + \sigma_2 P_2(\cos\theta) = \frac{m\omega}{3\pi c^2 R^2} - \frac{5m\omega}{6\pi c^2 R^2}P_2(\cos\theta) = \frac{m\omega}{3\pi c^2 R^2}(1 - \frac{5}{2}P_2(\cos\theta))$$

(b) Multipole moments $q_{\ell m}$ have contributions from both volume and surface charge distributions:

$$q_{\ell m} = \int Y^*_{\ell m}(\theta,\phi) r^\ell \rho d\tau + \oint Y^*_{\ell m}(\theta,\phi)R^\ell \sigma(\theta) da$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\left\{\rho\int_0^R r^{\ell+2}dr\int_{-1}^{+1} P_\ell^m(\cos\theta)d\cos\theta + R^\ell\int_{-1}^{+1}P_\ell^m(\cos\theta)\sigma(\theta)d\cos\theta\right\}\int_0^{2\pi} e^{im\phi}d\phi$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\left\{\rho\int_0^R r^{\ell+2}dr\int_{-1}^{+1} P_\ell(\cos\theta)d\cos\theta + R^\ell\int_{-1}^{+1}P_\ell(\cos\theta)\sigma(\theta)d\cos\theta\right\}2\pi\delta_{m,0}$$

Since $\sigma(\theta)$ is even in $\cos\theta$, the integral over $\cos\theta$ vanishes for odd ℓ values. Furthermore, the monopole moment also vanishes as a result of zero net charge on the sphere. Therefore, quadrupole moments are the lowest order non-vanishing moments. The quadrupole moment tensor has contributions from both the volume and the surface charge distributions:

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho d\tau + \oint (3x_i x_j - R^2\delta_{ij})\sigma(\theta) da$$

Due to the symmetry in ϕ and in z, the only non-vanishing components are Q_{11}, Q_{22} and Q_{33}.

$$Q_{33} = \rho\int(3z^2 - r^2)d\tau + R^2\int(3z^2 - R^2)\sigma(\theta)d\Omega$$

$$= 2\pi\rho\int_0^R r^4 dr\int_{-1}^{+1}(3\cos^2\theta - 1)d\cos\theta + 2\pi R^4\int_{-1}^{+1}(3\cos^2\theta - 1)\sigma(\theta)d\cos\theta$$

$$= 4\pi R^4\frac{m\omega}{3\pi c^2 R^2}\int_{-1}^{+1}P_2(\cos\theta)\{1 - \frac{5}{2}P_2(\cos\theta)\}d\cos\theta = -\frac{4m\omega R^2}{3c^2}$$

The $\phi-$symmetry of charge distribution and the fact that the tensor is traceless lead to:

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33} = \frac{2m\omega R^2}{3c^2}$$

(d) The electromotive force

$$\mathcal{E} = \int \vec{E}\cdot d\vec{\ell} = -\int \vec{v}\times\vec{B}\cdot d\vec{\ell} = \int_0^{\pi/2} \frac{2}{3}\mu_0 M(R\cos\theta\vec{\omega} - \omega\vec{r})\cdot(Rd\theta\hat{\theta})$$

$$= \frac{2}{3}\mu_0 M\omega R^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta = \frac{1}{3}\mu_0 M\omega R^2 = \frac{\mu_0 m\omega}{4\pi R}$$

Additional stuff for my record

The potential due to the surface charge

$$\Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\theta')}{|\vec{r}-\vec{r}'|}da' = \frac{R^2}{2\epsilon_0}\int \sigma(\theta')\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta')P_\ell(\cos\theta)d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)\int P_\ell(\cos\theta')\left\{1 - \frac{5}{2}P_2(\cos\theta')\right\}d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\left\{\frac{2}{r_>} - \frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\} = \frac{\mu_0}{3\pi}m\omega\left\{\frac{1}{r_>} - \frac{r_<^2}{r_>^3}\frac{P_2(\cos\theta)}{2}\right\}$$

The total potential outside the sphere ($r_< = a$ and $r_> = r$):

$$\Phi(\vec{r}) = \Phi_\rho(\vec{r}) + \Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_\rho}{r} + \frac{R^2\sigma_0}{\epsilon_0 r} - \frac{a^4\sigma_0}{2\epsilon_0 r^3}P_2(\cos\theta) = -\frac{m\omega R^2}{6\pi c^2\epsilon_0}\frac{P_2(\cos\theta)}{r^3} = -\frac{\mu_0}{6\pi}\frac{R^2}{r^3}P_2(\cos\theta)$$

The electric field outside the sphere:

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) = -\frac{\mu_0}{4\pi}\frac{m\omega R^2}{r^4}\left\{(3\cos^2\theta - 1)\hat{r} + 2\sin\theta\cos\theta\hat{\theta}\right\}$$

The total potential inside the sphere ($r_< = r$ and $r_> = a$):

$$\Phi(\vec{r}) = \Phi_\rho(\vec{r}) + \Phi_\sigma(\vec{r}) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2) + \frac{a\sigma_0}{\epsilon_0} - \frac{\sigma_0 r^2}{2\epsilon_0 a}P_2(\cos\theta)$$

$$= -\frac{m\omega}{6\pi c^2\epsilon_0 R}\left\{1 - \frac{r^2}{R^2}(1 - P_2(\cos\theta))\right\} = -\frac{\mu_0}{6\pi}\frac{m\omega}{R}\left\{1 - \frac{r^2}{R^2}(1 - P_2(\cos\theta))\right\}$$

Problem 6.8

In an external uniform electric field $\vec{E}$, the sphere is uniformly polarized with the polarization given by Eq. (4.57):

$$\vec{P} = 3\epsilon_0\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\vec{E}_0$$

Therefore, the bound volume and surface charge densities are:

$$\rho_b = -\nabla\cdot\vec{P} = 0, \qquad \sigma_b = \vec{P}\cdot\vec{n}$$

where $\vec{n}$ is the normal vector on the sphere surface. Since the sphere is rotating, the bound surface charge results an effective surface current with density:

$$\vec{K}_M = \sigma_b\vec{v} = (\vec{P}\cdot\vec{n})(\vec{\omega}\times\vec{r})|_{r=a} = a(\vec{P}\cdot\vec{n})(\omega\times\vec{n})$$

Comparing with the effective surface current density $\vec{K}_M = \vec{M} \times \vec{n}$ due to magnetization $\vec{M}$, we identify $a(\vec{P} \cdot \vec{n})\vec{\omega}$ as an effective magnetization. Therefore, the effective magnetic surface charge density

$$\sigma_M(\theta, \phi) = \vec{M}_{\text{eff.}} \cdot \vec{n} = a(\vec{P} \cdot \vec{n})(\vec{\omega} \cdot \vec{n}) = 3\epsilon_0 a \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} (\vec{E}_0 \cdot \vec{n})(\omega \cos\theta) = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a\omega E_0 \sin\theta \cos\theta \cos\phi$$

The magnetic scalar potential $\Phi_M(\vec{r})$ (Eq. (5.100)):

$$\Phi_M(\vec{r}) = \frac{1}{4\pi} \oint \frac{\sigma_M}{|\vec{r} - \vec{r'}|} da' = \frac{3}{4\pi} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3 \omega E_0 \int \frac{\sin\theta' \cos\theta' \cos\phi'}{|\vec{r} - \vec{r'}|} d\Omega'$$

Using the identity:

$$\sin\theta' \cos\theta' \cos\phi' = -\sqrt{\frac{8\pi}{15}} Re\left\{Y_{21}(\theta', \phi')\right\}$$

and expanding $1/|\vec{r} - \vec{r'}|$ using spherical harmonics, the integral becomes:

$$\int \frac{\sin\theta' \cos\theta' \cos\phi'}{|\vec{r} - \vec{r'}|} d\Omega' = -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') Y_{21}(\theta', \phi') d\Omega'\right\}$$

$$= -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \delta_{\ell,2} \delta_{m,1}\right\}$$

$$= \frac{4\pi}{5} \frac{r_<^2}{r_>^3} \left\{-\sqrt{\frac{8\pi}{15}} Re\{Y_{21}(\theta, \phi)\}\right\}$$

$$= \frac{4\pi}{5} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \cos\phi$$

where $r_< = \min(r, a)$ and $r_> = \max(r, a)$. Therefore, the scalar potential

$$\Phi_M(\vec{r}) = \frac{1}{4\pi} \oint \frac{\sigma_M}{|\vec{r} - \vec{r'}|} da' = \frac{3}{4\pi} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3 \omega E_0 \left\{\frac{4\pi}{5} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \cos\phi\right\}$$

$$= \frac{3}{5} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \omega E_0 \frac{a^3 r_<^2}{r^2 r_>^3} (r \sin\theta \cos\phi)(r \cos\theta)$$

Note

$$\frac{a^3 r_<^2}{r^2 r_>^3} = \frac{a^3 r_<^2 r_>^2}{r^2 r_>^5} = \frac{a^3 r^2 a^2}{r^2 r_>^5} = \left\{\frac{a}{r_>}\right\}^5$$

The magnetic field $\vec{H}$ can be determined from $\Phi_M(\vec{r})$:

$$\Phi_M(\vec{r}) = \frac{3}{5} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \omega E_0 \left\{\frac{a}{r_>}\right\}^5 \cdot xz = \frac{1}{5} P\omega \left\{\frac{a}{r_>}\right\}^5 \cdot xz$$

What if the electric field is along the rotational axis?

The effective magnetic surface charge density:

$$\sigma_M(\theta, \phi) = a(\vec{P} \cdot \vec{n})(\vec{\omega} \cdot \vec{n}) = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a\omega E_0 \cos^2\theta$$

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a^3\omega\epsilon_0 E_0\left\{\frac{1}{r_>}+\frac{2}{5}\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\}$$

Problem 6.11

(a) The momentum conservation equation

$$\frac{d}{dt}(\vec{P}_{\text{fields}}+\vec{P}_{\text{mech.}}) = \oint\sum_j T_{ij}n_j da = -\oint(-\vec{\vec{T}})\cdot\vec{n}da$$

implies that the projection of the momentum flow along the direction of $\vec{n}$ is given by $-\vec{\vec{T}}\cdot\vec{n}$ where $\vec{\vec{T}}$ is the Maxwell stress (momentum) tensor:

$$T_{ij} = \epsilon_0(E_iE_j - \frac{1}{2}E^2\delta_{ij}) + \mu_0(H_iH_j - \frac{1}{2}H^2\delta_{ij})$$

Physically $-T_{ij}$ is the rate at which the i^{th}-component of the momentum is crossing a unit area in the $j^{th}-$direction. In a Cartesian coordinate system with the $z-$axis along the wave propagation direction and $\vec{E}$ along the $x-$direction:

$$\vec{E} = E\hat{x}, \quad \vec{H} = H\hat{y}$$

The i^{th} component of the linear momentum flowing into the surface (in the direction $\vec{n}=\hat{z}$) per unit time per unit cross section is therefore

$$p_i = \sum_j(-T_{ij})n_j = -T_{i3} = -\epsilon_0(E_iE_3 - \frac{1}{2}E^2\delta_{i,3}) - \mu_0(H_iH_3 - \frac{1}{2}H^2\delta_{i,3}) = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)\delta_{i,3}$$

In the chosen coordinate system, the only non-vanishing component is p_z. The force exerted on the wave from the surface per unit area (according to Newton's second law):

$$F_z = \Delta p_z = (0-p_z) = -\frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

Therefore, the radiation pressure on the surface (Newton's third law):

$$P_z = -F_z = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

which is the energy density in the electromagnetic wave.

Problem 6.13

(a) Note: only need to work out the first non-zero terms in electric/magnetic fields. To a good approximation, the conductors are at equipotential and have uniform surface charge distributions. Choose a Cartesian coordinate system with its origin at the center of the capacitor, the $x-$axis parallel to the edge a and pointing to the current feed, the $y-$ axis perpendicular to the two planes. Let $Q(t) = Q_0e^{-i\omega t}$ be the total charge on the bottom plate, the electric field in between the plates is therefore

$$\vec{E}(\vec{r},t) = \frac{\sigma(t)}{\epsilon_0}\hat{y} = \frac{1}{\epsilon_0}\frac{Q_0}{ab}e^{-i\omega t}\hat{y}$$

The charge on the $x' < x$ portion of the bottom plate is:

$$Q(\vec{r},t) = b(x+\frac{a}{2})\sigma(t) = \frac{Q_0}{ab}e^{-i\omega t}b(x+\frac{a}{2}) = (\frac{1}{2}+\frac{x}{a})Q_0e^{-i\omega t}$$

The surface current density

$$\vec{K}(\vec{r},t) = -\frac{1}{b}\frac{\partial Q(\vec{r},t)}{\partial t}\hat{x} = i\omega(\frac{1}{2}+\frac{x}{a})\frac{Q_0}{b}e^{-i\omega t}\hat{x}$$

Note that K is maximum at $x = a/2$ and zero at $x = -a/2$ as expected. In between, the conduction current looses its strength to the displacement current. (b) The electric energy density and energy

$$W_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{1}{4}\epsilon_0|E|^2 = \frac{1}{4\epsilon_0}\frac{Q_0^2}{(ab)^2} \quad \Rightarrow \quad \int W_e d\tau = \frac{Q_0^2}{4\epsilon_0}\frac{d}{ab}$$

The magnetic energy density and energy

$$W_m = \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|B|^2 = \frac{\mu_0}{4}\omega^2(\frac{1}{2}+\frac{x}{a})^2\frac{Q_0^2}{b^2} \quad \Rightarrow \quad \int W_m d\tau = \frac{\mu_0\omega^2 adQ_0^2}{12b}$$

The input current

$$I(t) = \frac{dQ(t)}{dt} = -i\omega Q_0 e^{-i\omega t}$$

The reactance

$$X = \frac{4\omega}{|I|^2}\int(W_m - W_e)d\tau = \frac{4\omega}{\omega^2 Q_0^2}\left\{\frac{\mu_0\omega^2 adQ_0^2}{12b} - \frac{Q_0^2 d}{4\epsilon_0 ab}\right\} = \frac{\mu_0\omega ad}{3b} - \frac{d}{\epsilon_0\omega ab} = \omega L - \frac{1}{\omega C}$$

where $L = \mu_0 ad/3b$ and $C = \epsilon_0 ab/d$. Therefore, X is equivalent to the reactance of a capacitor C connecting in series with an inductor L.