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Physics 505: Solutions to Assignment #8

Problem 5.10
Useful identities:

$$\frac{\partial J_1(k\rho)}{\partial\rho} = \frac{k}{2}\{J_0(k\rho) - J_2(k\rho)\}; \quad \frac{\partial I_1(k\rho)}{\partial\rho} = \frac{k}{2}\{I_0(k\rho) + I_2(k\rho)\}; \quad \frac{\partial K_1(k\rho)}{\partial\rho} = -\frac{k}{2}\{K_0(k\rho) + K_2(k\rho)\}$$

$$\int_0^\infty dkk\cos(kz)K_1(ka) = \frac{a\pi}{2(a^2+z^2)^{3/2}}; \quad \int_0^\infty dkke^{-k|z|}J_1(ka) = \frac{a}{(a^2+z^2)^{3/2}}$$

The vector potential $\vec{A}$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{\ell'}}{|\vec{r}-\vec{r'}|} = \frac{\mu_0 Ia}{4\pi}\int_0^{2\pi}\frac{\hat{\phi}'}{|\vec{r}-\vec{r'}|}d\phi'$$

where $\hat{\phi}'$ is the unit vector along the ϕ' direction:

$$\hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

Consequently

$$\vec{A}(\vec{r}) = \frac{\mu_0 Ia}{4\pi}\int_0^{2\pi}\frac{-\sin\phi'\hat{x}+\cos\phi'\hat{y}}{|\vec{r}-\vec{r'}|}d\phi'$$

(a) Using the expansion of Eq. (3.148) with $z'=0$:

$$\frac{1}{|\vec{r}-\vec{r'}|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 Ia}{4\pi}\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)\int_0^{2\pi}e^{im(\phi-\phi')}(-\sin\phi'\hat{x}+\cos\phi'\hat{y})d\phi'$$

where $\rho_< = \min(a,\rho)$ and $\rho_> = \max(a,\rho)$. The integrals

$$\int_0^{2\pi}e^{im(\phi-\phi')}\sin\phi'd\phi' = Im\left\{e^{im\phi}\int_0^{2\pi}e^{i(1-m)\phi'}d\phi'\right\} = Im\left\{e^{im\phi}2\pi\delta_{m,1}\right\} = 2\pi\sin\phi\delta_{m,1}$$

$$\int_0^{2\pi}e^{im(\phi-\phi')}\cos\phi'd\phi' = Re\left\{e^{im\phi}\int_0^{2\pi}e^{i(1-m)\phi'}d\phi'\right\} = Re\left\{e^{im\phi}2\pi\delta_{m,1}\right\} = 2\pi\cos\phi\delta_{m,1}$$

Noting $(-\sin\phi\hat{x}+\cos\phi\hat{y}) = \hat{\phi}$, the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0 Ia}{4\pi}\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)(2\pi\delta_{m,1})(-\sin\phi\hat{x}+\cos\phi\hat{y})$$

$$= \frac{\mu_0 Ia}{\pi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)dk\hat{\phi}$$

i.e.

$$A_\phi(\rho,z) = \frac{\mu_0 Ia}{\pi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

(b) Using the expansion of Prob. 3.16(b) and noting $\rho' = a, z' = 0$:

$$\frac{1}{|\vec{r}-\vec{r'}|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(ka) e^{-k|z|}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(k\rho) J_m(ka) e^{-k|z|} \int_0^{2\pi} e^{im(\phi-\phi')}(-\sin\phi'\hat{x} + \cos\phi'\hat{y}) d\phi'$$

$$= \frac{\mu_0 I a}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(k\rho) J_m(ka) e^{-k|z|} (2\pi\delta_{m,1})(-\sin\theta\hat{x} + \cos\theta\hat{y})$$

$$= \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(k\rho) J_1(ka) e^{-k|z|} \hat{\phi}$$

i.e.,

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho)$$

(c)

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\hat{z} = B_\rho \hat{\rho} + B_z \hat{z}$$

Expansion of (a):

$$B_\rho(\rho, z) = -\frac{\partial A_\phi}{\partial z} = \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \sin(kz) I_1(k\rho_<) K_1(k\rho_>)$$

$$B_z(\rho, z) = \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} \{I_1(k\rho_<) K_1(k\rho_>)\}$$

On the $z-$axis ($\rho = 0$), $\rho_< = 0$ and $\rho_> = a$:

$$B_\rho(0, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \sin(kz) I_1(0) K_1(ka) = 0$$

$$B_z(0, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) \lim_{\rho\to 0} \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} \{I_1(k\rho) K_1(ka)\}$$

$$= \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \cos(kz) K_1(ka)$$

$$= \frac{\mu_0 I a}{\pi} \frac{a\pi}{2} \frac{1}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z^2)^{3/2}}$$

Expansion of (b):

$$B_\rho(\rho, z) = -\frac{\partial A_\phi}{\partial \rho} = \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(ka) J_1(k\rho) \frac{\partial}{\partial z} e^{-k|z|}$$

$$B_z(\rho, z) = \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi = \frac{\mu_0 I a}{2} \int_0^\infty e^{-k|z|} J_1(ka) \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} J_1(k\rho)$$

On the $z-$axis, $\rho = 0$:

$$B_\rho(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1(ka) J_1(0)\frac{\partial}{\partial z}e^{-k|z|} = 0$$

$$B_z(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk e^{-k|z|} J_1(ka) \lim_{\rho\to 0}\left\{\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right\} J_1(k\rho)$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk k e^{-k|z|} J_1(ka)$$

$$= \frac{\mu_0 I a^2}{2}\frac{1}{(a^2+z^2)^{3/2}}$$

Problem 5.20

Useful identity:

$$\int_V (\vec{C}\cdot\nabla)\vec{D}d\tau = -\int_V (\nabla\cdot\vec{C})\vec{D}d\tau + \oint(\vec{C}\cdot\vec{n})\vec{D}da$$

(a) The force on the bound volume and surface currents

$$\vec{F} = \int_V (\nabla\times\vec{M})\times\vec{B}_e d\tau + \oint(\vec{M}\times\vec{n})\times\vec{B}_e da$$

Using the product rule:

$$(\nabla\times\vec{M})\times\vec{B}_e = -\vec{B}_e\times(\nabla\times\vec{M}) = (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} + \vec{M}\times(\nabla\times\vec{B}_e) - \nabla(\vec{M}\cdot\vec{B}_e)$$

Since $\vec{B}_e$ is an external field, we have $\nabla\times\vec{B}_e = 0$ in the region of interest. So

$$(\nabla\times\vec{M})\times\vec{B}_e = (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} - \nabla(\vec{M}\cdot\vec{B}_e)$$

Also

$$(\vec{M}\times\vec{n})\times\vec{B}_e = -\vec{B}_e\times(\vec{M}\times\vec{n}) = -(\vec{B}_e\cdot\vec{n})\vec{M} + (\vec{B}_e\cdot\vec{M})\vec{n}$$

The force

$$\vec{F} = \int_V\left\{\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} - \nabla(\vec{M}\cdot\vec{B}_e)\right\}d\tau + \oint\left\{(\vec{B}_e\cdot\vec{M})\vec{n} - (\vec{B}_e\cdot\vec{n})\vec{M}\right\}da$$

$$= \int_V\left\{\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M}\right\}d\tau - \oint\left\{(\vec{B}_e\cdot\vec{n})\vec{M}\right\}da$$

Using the identity

$$\int_V (\vec{C}\cdot\nabla)\vec{D}d\tau = -\int_V (\nabla\cdot\vec{C})\vec{D}d\tau + \oint(\vec{C}\cdot\vec{n})\vec{D}da$$

on the first two integrals:

$$\vec{F} = -\int_V(\nabla\cdot\vec{M})\vec{B}_e d\tau + \oint(\vec{M}\cdot\vec{n})\vec{B}_e da - \int_V(\nabla\cdot\vec{B}_e)\vec{M}d\tau + \oint(\vec{B}_e\cdot\vec{n})\vec{M}da - \oint(\vec{B}_e\cdot\vec{n})\vec{M}da$$

$$= -\int_V(\nabla\cdot\vec{M})\vec{B}_e d\tau + \oint(\vec{M}\cdot\vec{n})\vec{B}_e da$$

Problem 5.22

Since there is no free current, the magnetic scalar potential approach is applicable. Within the infinite permeable medium, the auxiliary field $\vec{H}$ vanishes. Since $\vec{n} \times \vec{H}$ is continuous at the surface, $\vec{H}$ outside the medium must be perpendicular to the surface. Consequently, the surface is at equipotential and we can set its potential to be zero. Therefore, this problem can be treated like the analogous electrostatic problems with a polarized bar against a perfect conductor at zero potential. We can use the method of image and replace the medium with an image bar magnet with magnetization $-\vec{M}$. From the result of Prob. 5.20, the force on the bar magnet is given by

$$\vec{F} = \int_V \rho_M \vec{B} d\tau + \oint \sigma_M \vec{B} da$$

where $\vec{B}$ is the field due to the image bar magnet. ρ_M, σ_M are the effective volume and surface magnetic charge densities respectively:

$$\rho_M = -\nabla \cdot \vec{M} = 0, \qquad \sigma_M = \vec{M} \cdot \vec{n}$$

In a coordinate system with its origin at the joint between the bar and the image and its $z-$axis along the $\vec{M}$ direction, the magnetic field due to the image bar along the $z-$axis is given by (Prob. 5.19, shifting $z-$axis by $L/2$):

$$\vec{B}(z) = -\frac{1}{2}\mu_0 M \left\{ \frac{z}{\sqrt{a^2+z^2}} - \frac{z+L}{\sqrt{a^2+(z+L)^2}} \right\} \hat{z}$$

where a is the effective radius of the bar ($\pi a^2 = A$): Assuming the bar is long and narrow, the force is then

$$\vec{F} = \oint (\vec{M} \cdot \vec{n}) \vec{B} da \approx AM\{B(L) - B(0)\} \hat{z}$$

$$= -\mu_0 A M^2 L \left\{ \frac{1}{\sqrt{a^2+L^2}} - \frac{1}{\sqrt{a^2+4L^2}} \right\} \hat{z} \approx -\frac{1}{2}\mu_0 A M^2 \hat{z}$$

The force points to $-\hat{z}$ direction, *i.e.*, the bar is attracted to the medium.

Problem 5.27

From the Ampere's law, we can calculate the magnetic fields in the three regions:

$$B(\rho) = \frac{\mu_c I}{\pi b^2}\rho \quad (0 < \rho < b); \quad B(\rho) = \frac{\mu_0 I}{2\pi\rho} \quad (b < \rho < a); \quad B(\rho) = 0 \quad (\rho > a)$$

where μ_c is the permeability of the conductor. The $\vec{B}$ is along the $\hat{\phi}$. The magnetic energy per unit length

$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_c}\int_0^b B^2 \rho d\rho d\phi + \frac{1}{2\mu_0}\int_b^a B^2 \rho d\rho d\phi$$

$$= \frac{1}{2\mu_c} \cdot 2\pi \int_0^b \left\{ \frac{\mu_c I}{2\pi b^2}\rho \right\}^2 \rho d\rho + \frac{1}{2\mu_0} \cdot 2\pi \int_b^a \left\{ \frac{\mu_0 I}{2\pi\rho} \right\}^2 \rho d\rho$$

$$= \frac{I^2}{4\pi}\left\{ \frac{\mu_c}{4} + \mu_0 \ln\frac{a}{b} \right\}$$

The self-inductance L per unit length

$$L = \frac{1}{4\pi}\left\{ \frac{\mu_c}{4} + \mu_0 \ln\frac{a}{b} \right\}$$

For the case of $\mu_c = \mu_0$, it simplifies to:

$$L = \frac{\mu_0}{4\pi}\left\{\frac{1}{4} + \ln\frac{a}{b}\right\}$$

If the inner conductor is a thin hollow tube, there will be no magnetic field inside the hollow tube. Consequently the self-inductance per unit length would be:

$$L = \frac{\mu_0}{4\pi}\ln\frac{a}{b}$$

Problem 5.29

Inside an ideal conductor, the electric and magnetic fields vanish. Furthermore, all charges and currents are on the conductor surfaces. In a cylindrical coordinate system with the $z-$axis along the direction of the two conductors, the surface charge and current densities are $z-$independent. Let $\sigma_1(\rho,\phi)$ and $\sigma_2(\rho,\phi)$ be the surface charge densities on the two conductors, the surface current densities are then $\sigma_1(\rho,\phi)v$ and $\sigma_2(\rho,\phi)v$ along the $z-$direction, where v is the speed of the charges. Consequently, the scalar and vector potentials are:

$$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon}\left\{\int\frac{\sigma_1(\rho',\phi')}{|\vec{r}-\vec{r'}|}da_1' + \int\frac{\sigma_2(\rho',\phi')}{|\vec{r}-\vec{r'}|}da_2'\right\}$$

$$A_z(\rho,\phi) = \frac{\mu}{4\pi}\left\{\int\frac{\sigma_1(\rho',\phi')v}{|\vec{r}-\vec{r'}|}da_1' + \int\frac{\sigma_2(\rho',\phi')v}{|\vec{r}-\vec{r'}|}da_2'\right\} = \mu\epsilon v\Phi(\rho,\phi)$$

where $\int da_1'$ and $\int da_2'$ integrate over the surfaces of the two conductors. Note that $A_\rho = A_\phi = 0$. Let $\pm q$ be the total charge per unit length on the two conductors, then we have $qv = I$. Therefore,

$$A_z(\rho,\phi) = \mu\epsilon\frac{I}{q}\Phi(\rho,\phi)$$

The electric and magnetic fields are

$$E^2 = \{-\nabla\Phi\}^2 = \left\{\frac{\partial\Phi}{\partial\rho}\right\}^2 + \left\{\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right\}^2$$

$$B^2 = |\nabla\times\vec{A}|^2 = \left\{\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\right\}^2 + \left\{\frac{\partial A_z}{\partial\rho}\right\}^2 = \left\{\mu\epsilon\frac{I}{q}\right\}^2 E^2$$

The energies in electric and magnetic fields per unit length are:

$$W_E = \frac{1}{2}\frac{q^2}{C} = \frac{\epsilon}{2}\int E^2 d\tau$$

$$W_B = \frac{1}{2}LI^2 = \frac{1}{2\mu}\int B^2 d\tau = \frac{1}{2}\mu\epsilon^2\frac{I^2}{q^2}\int E^2 d\tau$$

Taking the ratio of W_B and W_E leads to the final result:

$$LC = \mu\epsilon$$