Instructor: Jianming Qian
Due date: November 1, 2001

Physics 505: Solutions to Assignment #7

Problem 5.6

Using the principle of superposition, the magnetic field at $\vec{r}$ in the cavity is equal to that of a conductor without the hole minus that of a smaller conductor filling the hole with the same volume current densities, *i.e.*:

$$\vec{B}_{\text{cavity}}(\vec{r}) = \vec{B}_{\text{nohole}}(\vec{r}) - \vec{B}_{\text{hole}}(\vec{r})$$

In the polar coordinate system with the $z-$axis along the cylinder axis, $\vec{B}_{\text{nohole}}$ can be calculated from Ampere's law:

$$\vec{B}_{\text{nohole}}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} = \frac{\mu_0 \pi\rho^2 J}{2\pi\rho}\hat{\phi} = \frac{1}{2}\mu_0 J\rho\hat{\phi}$$

where $\hat{\phi}$ is the unit vector along the $\phi-$direction and J is the volume current density. Similarly, we have

$$\vec{B}_{\text{hole}}(\vec{r}) = \frac{1}{2}\mu_0 J\rho'\hat{\phi}'$$

where ρ' and ϕ' are measured with respect to the axis of the hole. Therefore,

$$\vec{B}_{\text{cavity}}(\vec{r}) = \frac{1}{2}\mu_0 J(\rho\hat{\phi} - \rho'\hat{\phi}')$$

Let $\vec{d}$ be the vector from the cylinder axis to the hole axis, we have:

$$\vec{\rho} - \vec{\rho'} = \vec{d}$$

Cross multiplying the above equation with $\hat{z}$ (the unit vector along the $z-$direction) and noting $\hat{z}\times\hat{\rho} = \hat{\phi}$, we get:

$$\rho\hat{\phi} - \rho'\hat{\phi}' = \hat{z}\times\vec{d}$$

Consequently, the field inside the cavity:

$$\vec{B}_{\text{cavity}}(\vec{r}) = \frac{1}{2}\mu_0 J\hat{z}\times\vec{d}$$

The field is uniform and in the direction perpendicular to the line joinning the axes of the cylinder and the hole. In terms of the current I on the cylinder:

$$J = \frac{I}{\pi(a^2-b^2)}; \qquad \Rightarrow \qquad \vec{B}_{\text{cavity}}(\vec{r}) = \frac{\mu_0 I}{2\pi(a^2-b^2)}\hat{z}\times\vec{d}$$

Problem 5.13

The rotating surface charges result surface currents. In the spherical coordinate system with the $z-$axis along the rotation axis, the surface current density

$$\vec{K}(\vec{r}) = \sigma\vec{v}(\vec{r}) = \sigma\omega a\sin\theta\hat{\phi}$$

where $\hat{\phi}$ is the unit vector along the $\phi-$direction. Therefore, the vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint\frac{\vec{K}(\vec{r'})}{|\vec{r}-\vec{r'}|}da' = \frac{1}{4\pi}\mu_0\sigma\omega a^3\int\frac{\sin\theta'\hat{\phi}'}{|\vec{r}-\vec{r'}|}d\Omega'$$

To carry out the above integral, we project $\hat{\phi}'$ along fixed $x-$ and $y-$directions and expand $1/|\vec{r}-\vec{r'}|$ in spherical harmonics.

$$\hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

where $r_< = \min(a, r)$ and $r_> = \max(a, r)$. The vector potential is therefore:

$$\vec{A}(\vec{r}) = \mu_0 \omega \sigma a^3 \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi')(-\sin\theta' \sin\phi' \hat{x} + \sin\theta' \cos\phi' \hat{y}) d\Omega'$$

Note that $\sin\theta' \sin\phi'$ and $\sin\theta' \cos\phi'$ can be written in terms of $Y_{1,1}$:

$$Y_{1,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}; \quad \Rightarrow \quad \sin\theta' \sin\phi' = -\sqrt{\frac{8\pi}{3}} Im\{Y_{1,1}(\theta', \phi')\}; \quad \sin\theta' \cos\phi' = -\sqrt{\frac{8\pi}{3}} Re\{Y_{1,1}(\theta', \phi')\}$$

and the integrals can be carried out using the orthogonality properties of spherical harmonics:

$$Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') \sin\theta' \sin\phi' d\Omega' = -\sqrt{\frac{8\pi}{3}} Im \left\{ Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') Y_{1,1}(\theta', \phi') d\Omega' \right\} = \sin\theta \sin\phi \delta_{\ell,1} \delta_{m,1}$$

$$Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') \sin\theta' \cos\phi' d\Omega' = -\sqrt{\frac{8\pi}{3}} Re \left\{ Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') Y_{1,1}(\theta', \phi') d\Omega' \right\} = \sin\theta \cos\phi \delta_{\ell,1} \delta_{m,1}$$

As results of the above, the vector potential is

$$\vec{A}(\vec{r}) = \mu_0 \omega \sigma a^3 \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \delta_{\ell,1} \delta_{m,1} \sin\theta(-\sin\phi \hat{x} + \cos\phi \hat{y}) = \frac{1}{3} \mu_0 \omega \sigma a^3 \sin\theta \frac{r_<}{r_>^2} \hat{\phi}$$

Inside the sphere, $r_< = r$ and $r_> = a$:

$$\vec{A}(\vec{r}) = \frac{1}{3} \mu_0 \omega \sigma a(-y\hat{x} + x\hat{y}) = \frac{1}{3} \mu_0 \omega \sigma a r \sin\theta \hat{\phi} = \frac{1}{3} \mu_0 \sigma a \ \vec{\omega} \times \vec{r}$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A} = \frac{2}{3} \mu_0 \omega \sigma a \hat{z} = \frac{2}{3} \mu_0 \sigma a \ \vec{\omega}$$

The field inside the sphere is uniform and point to $z-$direction. Outside the sphere, $r_< = a$ and $r_> = r$:

$$\vec{A}(\vec{r}) = \frac{1}{3} \mu_0 \omega \sigma a^4 \frac{\sin\theta}{r^2} \hat{\phi} = \frac{1}{3} \mu_0 \sigma a^4 \frac{\vec{\omega} \times \vec{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A} = \frac{1}{3} \mu_0 \omega \sigma a^4 \{ \frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \} = \frac{\mu_0}{4\pi} \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{r^3}$$

This is the field due to a magnetic dipole

$$\vec{m} = \frac{4}{3} \pi a^3 (\sigma a \ \vec{\omega})$$

Problem 5.18

(a) From the result of Prob. 5.17, the magnetic field at the current loop can be calculated by replacing the medium with an image current of magnitude

$$I' = \frac{\mu - \mu_0}{\mu + \mu_0} I$$

In a spherical coordinate system with its $x-y$ plane defined by the imagine current loop, its origin at the center of the loop and its $z-$axis pointing to the current loop I, the magnetic field due to the imagine current is given by Eq. (5.48) and (5.49). At the location of the current loop,

$$r_< = a, \quad r_> = r = \sqrt{a^2 + 4d^2}, \quad \cos\theta = \frac{2d}{\sqrt{a^2 + 4d^2}}$$

Therefore,

$$B_r = \frac{1}{2}\mu_0 I' a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \frac{a^{2n+1}}{(a^2+4d^2)^{n+3/2}} P_{2n+1}(\cos\theta)$$

$$B_\theta = -\frac{1}{4}\mu_0 I' a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \frac{a^{2n+1}}{(a^2+4d^2)^{n+3/2}} P^1_{2n+1}(\cos\theta)$$

The force on the current loop:

$$\vec{F} = \oint \vec{I} \times \vec{B} d\ell = \oint I\hat{\phi} \times (B_r \hat{r} + B_\theta \hat{\theta}) d\ell = \oint (IB_r \hat{\theta} - IB_\theta \hat{r}) d\ell$$

Note that both B_r and B_θ are constants of the integration and that

$$\oint \hat{\theta} d\ell = -\sin\theta \hat{z} \oint d\ell = -2\pi a \sin\theta \hat{z}, \qquad \oint \hat{r} d\ell = \cos\theta \hat{z} \oint d\ell = 2\pi a \cos\theta \hat{z}$$

The force acting on the loop:

$$\vec{F} = IB_r \oint \hat{\theta} d\ell - IB_\theta \oint \hat{r} d\ell = -2\pi a I (B_r \sin\theta + B_\theta \cos\theta)\hat{z}$$

$$= -\pi\mu_0 a^2 I I' \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n} \frac{a^{2n+1}}{(a^2+4d^2)^{n+2}} \left\{ \frac{a}{n!} P_{2n+1}(\cos\theta) - \frac{d}{(n+1)!} P^1_{2n+1}(\cos\theta) \right\} \hat{z}$$

(c) For $d \gg a$, the force is dominated by the $n = 0$ term:

$$\vec{F} = -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \{a P_1(\cos\theta) - d P^1_1(\cos\theta)\} \hat{z}$$

$$= -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \{a\cos\theta + d\sin\theta)\} \hat{z}$$

$$= -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \left\{ a \frac{2d}{\sqrt{a^2+4d^2}} + d \frac{a}{\sqrt{a^2+4d^2}} \right\} \hat{z}$$

$$= -3\pi\mu_0 I I' \frac{a^4 d}{(a^2+4d^2)^{5/2}} \hat{z} \approx -\frac{3}{32}\pi\mu_0 I^2 \frac{\mu-\mu_0}{\mu+\mu_0} \frac{a^4}{d^4} \hat{z}$$

The force is attractive. Alternatively, for $d \gg a$, both current loops can be approximated as point dipoles with dipole moments

$$\vec{m} = I(\pi a^2)\hat{z}, \qquad \vec{m'} = I'(\pi a^2)\hat{z}$$

The magnetic field at $\vec{m}$ due to $\vec{m'}$:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3\hat{z}(\hat{z} \cdot \vec{m'}) - \vec{m'}}{z^3}$$

where $z = \sqrt{a^2+4d^2}$ is the separation between the two dipoles. Consequently, the potential energy:

$$U = -\vec{m} \cdot \vec{B} = -\frac{\mu_0}{4\pi} \frac{3(\hat{z} \cdot \vec{m})(\hat{z} \cdot \vec{m'}) - \vec{m} \cdot \vec{m'}}{z^3} = -\frac{\mu_0}{2\pi} \frac{mm'}{z^3}$$

and the force

$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial z}|_{z=\sqrt{a^2+4d^2}}\ \hat{z} = -\frac{3\mu_0}{2\pi}\frac{mm'}{(a^2+4d^2)^2}\hat{z}$$

$$= -\frac{3}{2}\pi\mu_0 II'\frac{a^4}{(a^2+4d^2)^2}\hat{z} \approx -\frac{3}{32}\pi\mu_0 I^2\frac{\mu-\mu_0}{\mu+\mu_0}\frac{a^4}{d^4}\hat{z}$$

agrees with the result from (a).

Problem 5.19

There is no free current. Therefore, the scalar potential approach is applicable.

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\int\frac{-\nabla\cdot\vec{M}}{|\vec{r}-\vec{r'}|}d\tau' + \frac{1}{4\pi}\oint\frac{\vec{M}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|}da'$$

The effective magnetic volume charge $\rho_M = -\nabla\cdot\vec{M} = 0$. In the cylindrical coordinate system with its origin at the center of the cylinder and its $z-$axis along the axis of the cylinder in the magnetization direction,

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint\frac{\vec{M}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|}da' = \frac{1}{4\pi}\int_{\text{top}}\frac{M_0}{|\vec{r}-\vec{r'}|}da' - \frac{1}{4\pi}\int_{\text{bottom}}\frac{M_0}{|\vec{r}-\vec{r'}|}da'$$

Along the $z-$ axis,

$$\Phi_M(z) = \frac{M_0}{4\pi}(2\pi)\int_0^a d\rho'\left\{\frac{1}{\sqrt{\rho'^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho'^2+(z+L/2)^2}}\right\}$$

$$= \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - |z-\frac{L}{2}| - \sqrt{a^2+(z-\frac{L}{2})^2} + |z+\frac{L}{2}|\right\}$$

Therefore,

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} + L\right\} \qquad \text{for} \quad z > \frac{L}{2}$$

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} + 2z\right\} \qquad \text{for} \quad -\frac{L}{2} < z < \frac{L}{2}$$

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} - L\right\} \qquad \text{for} \quad z < -\frac{L}{2}$$

The auxiliary fields along the $z-$axis:

$$\vec{H}(z) = -\frac{\partial\Phi_M(z)}{\partial z}\hat{z} = -\frac{M_0}{2}\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\right\}\hat{z} \quad \text{inside the cylinder}$$

$$\vec{H}(z) = -\frac{\partial\Phi_M(z)}{\partial z}\hat{z} = -\frac{M_0}{2}\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{outside the cylinder}$$

The magnetic fields along the $z-$axis:

$$\vec{B}(z) = \mu_0(\vec{H}+\vec{M}) = -\frac{1}{2}\mu_0 M_0\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{inside the cylinder}$$

$$\vec{B}(z) = \mu_0\vec{H} = -\frac{1}{2}\mu_0 M_0\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{outside the cylinder}$$