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Physics 505: Solutions to Assignment #6

Problem 4.1
(a) The charge density

$$\rho(\vec{r}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\{\delta(\phi)+\delta(\phi-\frac{\pi}{2})-\delta(\phi-\pi)-\delta(\phi-\frac{3\pi}{2})\}$$

The multipole moment

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\vec{r'})d\tau' = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{\ell+m)!}}qa^{\ell}P^m_{\ell}(0)\{1+e^{-im\pi/2}-e^{-im\pi}-e^{-i3m\pi/2}\}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{\ell+m)!}}qa^{\ell}P^m_{\ell}(0)\left\{\{1-(-1)^m\}\{1+e^{-im\pi/2}\}\right\}$$

Since $P^m_\ell(x)$ is odd if $\ell+m$ =odd, $P^m_\ell(0)$ vanishes unless $\ell+m$ =even. Furthermore, $q_{\ell m}$ vanishes if m is even. Therefore, for non-vanishing $q_{\ell m}$, both ℓ and m must be odd. Let $\ell = 2j+1$ and $m = 2k+1$:

$$q_{\ell=2j+1,m=2k+1} = 2\{1+(-1)^{k+1}i\}qa^{\ell}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P^m_{\ell}(0)$$

$$= -\frac{1-(-1)^k i}{2^{\ell}\ell!}qa^{\ell}\sqrt{\frac{2\ell+1}{\pi}\frac{(\ell-m)!}{(\ell+m)!}}\left\{\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^{\ell}\right\}|_{x=0}$$

The first two sets of non-vanishing moments are

$$q_{1,\pm1} = \mp(1\mp i)\sqrt{\frac{3}{2\pi}}qa$$

$$q_{3,\pm1} = \pm(1\mp i)\sqrt{\frac{21}{16\pi}}qa^3$$

$$q_{3,\pm3} = \mp(1\pm i)\sqrt{\frac{35}{16\pi}}qa^3$$

(b) The charge density:

$$\rho(\vec{r}) = \frac{q}{2\pi r^2}\{\delta(r-a)\delta(\cos\theta-1)+\delta(r-a)\delta(\cos\theta+1)-\delta(r)\}$$

The multipole moment

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\vec{r'})d\tau' = \frac{q}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\int_0^{2\pi}e^{-im\phi'}d\phi'\{a^{\ell}P^m_{\ell}(1)+a^{\ell}P^m_{\ell}(-1)-2\delta_{\ell,0}\}$$

$$= q\sqrt{\frac{2\ell+1}{4\pi}}\{a^{\ell}P_{\ell}(1)+a^{\ell}P_{\ell}(-1)-2\delta_{\ell 0}\}\delta_{m0} = q\sqrt{\frac{2\ell+1}{4\pi}}\{a^{\ell}(1+(-1)^{\ell})-2\delta_{\ell 0}\}\delta_{m0}$$

The first two sets of non-vanishing moments are:

$$q_{2,0} = \sqrt{\frac{5}{\pi}}qa^2; \quad q_{2,m\neq 0} = 0$$

$$q_{4,0} = \sqrt{\frac{9}{\pi}} q a^4; \qquad q_{4,m\neq 0} = 0$$

(c) The potential

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{a^\ell\{1+(-1)^\ell\} - 2\delta_{\ell,0}}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{q}{2\pi\epsilon_0} \sum_{k=1}^{\infty} \frac{a^{2k}}{r^{2k+1}} P_{2k}(\cos\theta) = \frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} (3\cos^2\theta - 1) + ...$$

In the $x-y$ plane, $\cos\theta = 0$:

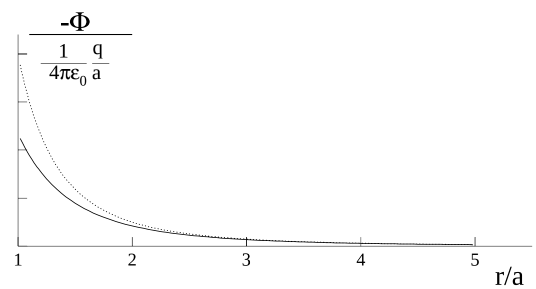
$$\Phi(r, \theta = \pi/2) = -\frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} + ...$$

(d) The exact potential in the $(x-y)$ plane

$$\Phi(r, \theta = \pi/2) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{2q}{\sqrt{r^2+a^2}} - \frac{2q}{r} \right\}$$

$$= \frac{q}{2\pi\epsilon_0} \frac{1}{r} \left\{ \frac{1}{\sqrt{1+(a/r)^2}} - 1 \right\} = -\frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} + ...$$

agrees with the result of (c).



The potential (in units of $(q/4\pi\epsilon_0 a)$) in $x-y$ plane as functions of r/a. The dotted line is the approximation from (c) and the solid line is the exact calculation of (d).

Problem 4.2
The potential at $\vec{r}$ due to a point dipole $\vec{p}$ at $\vec{r_0}$:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \frac{\vec{r}-\vec{r_0}}{|\vec{r}-\vec{r_0}|^3} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} \cdot \left\{ \vec{p}\delta^3(\vec{r'}-\vec{r_0}) \right\} d\tau' = \frac{1}{4\pi\epsilon_0} \int_V \nabla' \left\{ \frac{1}{|\vec{r}-\vec{r'}|} \right\} \cdot \left\{ \vec{p}\delta^3(\vec{r'}-\vec{r_0}) \right\} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left\{ \frac{\vec{p}\delta^3(\vec{r'}-\vec{r_0})}{|\vec{r}-\vec{r'}|} \right\} d\tau' - \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\vec{r}-\vec{r'}|} \nabla' \cdot \left\{ \vec{p}\delta^3(\vec{r'}-\vec{r_0}) \right\} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \oint \frac{\vec{p}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|} \delta^3(\vec{r'}-\vec{r_0}) da' + \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\vec{r}-\vec{r'}|} \left\{ -\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0}) \right\} d\tau'$$

where $\vec{r_0}$ is inside the volume V. Therefore, the surface integral vanishes:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\vec{r}-\vec{r'}|}\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\} d\tau'$$

which is the potential by an effective charge density

$$\rho_{\text{eff}}(\vec{r}) = -\vec{p}\cdot\nabla\delta^3(\vec{r}-\vec{r_0})$$

The energy of the dipole in an electric field:

$$W = -\vec{p}\cdot\vec{E}(\vec{r_0}) = \int_V \delta^3(\vec{r'}-\vec{r_0})\left\{-\vec{p}\cdot\vec{E}(\vec{r'})\right\} d\tau' = \int_V \delta^3(\vec{r'}-\vec{r_0})\vec{p}\cdot\nabla'\Phi(\vec{r'})d\tau'$$

$$= \int_V \left\{\nabla'\cdot\{\vec{p}\Phi(\vec{r'})\delta^3(\vec{r'}-\vec{r_0})\} - \Phi(\vec{r'})\nabla'\cdot\{\vec{p}\delta^3(\vec{r'}-\vec{r_0})\}\right\} d\tau'$$

$$= \oint(\vec{p}\cdot\vec{n'})\Phi(\vec{r'})\delta^3(\vec{r'}-\vec{r_0})da' + \int_V\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}\Phi(\vec{r'})d\tau'$$

Again the surface integral vanishes since $\vec{r_0} \in V$. Therefore,

$$W = \int_V\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}\Phi(\vec{r'})d\tau'$$

which is the energy of a distribution of charge density

$$\rho_{\text{eff}}(\vec{r}) = -\vec{p}\cdot\nabla\delta^3(\vec{r}-\vec{r_0})$$

Problem 4.10

(a) The electric fields in the two regions must be the same (otherwise, it will lead to different potential differences between the inner and the other spheres in the two regions). Applying Gauss's law in dielectrics on a Gaussian surface of radius r ($a < r < b$) and noting $\vec{D}$ is along the radial direction by symmetry

$$\oint \vec{D}\cdot\vec{n}da = Q; \quad \Rightarrow \quad (\epsilon E + \epsilon_0 E)2\pi r^2 = Q$$

Therefore, the electric field everywhere between the sphere is

$$\vec{E} = \frac{Q}{2\pi(\epsilon+\epsilon_0)}\frac{\vec{r}}{r^3}$$

(b) The free surface charge densities on the inner sphere are:

$$\sigma = \epsilon_0\vec{E}(r=a)_\perp = \frac{Q}{2\pi a^2}\frac{\epsilon_0}{\epsilon+\epsilon_0} \qquad \text{the region without the dielectric}$$

$$\sigma = \epsilon\vec{E}(r=a)_\perp = \frac{Q}{2\pi a^2}\frac{\epsilon}{\epsilon+\epsilon_0} \qquad \text{the region with the dielectric}$$

(c) The polarization in the region with the dielectric:

$$\vec{P} = (\epsilon-\epsilon_0)\vec{E} = \frac{Q}{2\pi}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{\vec{r}}{r^3}$$

Therefore, the polarization surface charge density

$$\sigma_b = \{\vec{P}\cdot\vec{n}\}_{r=a} = -P_r(r=a) = -\frac{Q}{2\pi a^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}$$

In the region without the dielectric, the polarization surface charge density $\sigma_b = 0$.

Problem 4.13

At equilibrium, the electrostatic force balances the gravity. For a fixed potential difference V, the electrostatic force is given by

$$F_e = \frac{dW}{dh} = \frac{1}{2}V^2\frac{dC}{dh}$$

where C is the total capacitance of the section above the liquid surface:

$$C = C_h + C_{\ell-h}$$

Here C_h is the capacitance of the section with the liquid in between the two electrodes and $C_{\ell-h}$ is the capacitance of the section above, ℓ is the height above the liquid surface. Note for a cylindrical capacitor in vacuum, the capacitance per unit length is

$$C_0 = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Therefore,

$$C = \frac{2\pi\epsilon_0(1+\chi_e)h}{\ln(b/a)} + \frac{2\pi\epsilon_0(\ell-h)}{\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a)}\{\chi_e h + \ell\}$$

$$F_e = \frac{1}{2}V^2\frac{dC}{dh} = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}, \qquad F_g = \rho\pi(b^2-a^2)hg$$

$$F_e = F_g \qquad \Rightarrow \qquad \chi_e = \frac{(b^2-a^2)\rho hg\ln(b/a)}{\epsilon_0 V^2}$$