Instructor: Jianming Qian
Due date: October 11, 2001

Physics 505: Solutions to Assignment #5

Problem 3.10

Useful identifies:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \tan^{-1}(x) \qquad \sum_{n=1}^{\infty} \frac{x^n}{n} = \ln(1-x)$$

(a) From the result of Problem 3.9:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi\rho}{L}) \sin(\frac{n\pi}{L} z)\{A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)\}$$

with

$$A_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z) \sin(m\phi) \sin(\frac{n\pi}{L} z)$$

$$= \frac{2}{\pi L I_m(n\pi b/L)} \int_0^L \sin(\frac{n\pi}{L} z) dz \left\{ \int_{-\pi/2}^{\pi/2} (+V) \sin(m\phi) d\phi + \int_{\pi/2}^{3\pi/2} (-V) \sin(m\phi) d\phi \right\}$$

$$= \frac{2}{\pi L I_m(n\pi b/L)} \frac{L}{n\pi} \{1 - (-1)^n\} \cdot 0 = 0$$

$$B_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z) \cos(m\phi) \sin(\frac{n\pi}{L} z)$$

$$= \frac{2}{\pi L I_m(n\pi b/L)} \int_0^L \sin(\frac{n\pi}{L} z) dz \left\{ \int_{-\pi/2}^{\pi/2} (+V) \cos(m\phi) d\phi + \int_{\pi/2}^{3\pi/2} (-V) \cos(m\phi) d\phi \right\}$$

$$= \frac{2}{\pi L I_m(n\pi b/L)} \frac{L}{n\pi} \{(-1)^n - 1\} \frac{V}{m} \{3 \sin(\frac{m\pi}{2}) - \sin(\frac{3m\pi}{2})\}$$

B_{mn}'s are non-vanishing if both m and n are odd numbers. Let $m = 2k+1$ and $n = 2\ell + 1$:

$$B_{2k+1, 2\ell+1} = \frac{16(-1)^{k+1} V}{(2k+1)(2\ell+1)\pi^2} \frac{1}{I_{2k+1}\{(2\ell+1)\pi b/L\}}$$

Therefore, the potential

$$\Phi(\rho, \phi, z) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{16(-1)^{k+1} V}{(2k+1)(2\ell+1)\pi^2} \frac{I_{2k+1}\{(2\ell+1)\pi\rho/L\}}{I_{2k+1}\{(2\ell+1)\pi b/L\}} \sin\{\frac{(2\ell+1)\pi}{L} z\} \cos\{(2k+1)\phi\}$$

(b) $L \gg b \Rightarrow \pi\rho/L \ll 1$ and $\pi b/L \ll 1$.

$$\frac{I_{2k+1}\{(2\ell+1)\pi\rho/L\}}{I_{2k+1}\{(2\ell+1)\pi b/L\}} \Rightarrow (\frac{\rho}{b})^{2k+1}$$

The potential at $z = L/2$:

$$\Phi(\rho, \phi, z) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{16(-1)^{k+1} V}{(2k+1)(2\ell+1)\pi^2} (\frac{\rho}{b})^{2k+1} (-1)^\ell \cos\{(2k+1)\phi\}$$

$$= \left\{ \sum_{k=0}^{\infty} \frac{16(-1)^{k+1}V}{(2k+1)\pi^2 L} (\frac{\rho}{b})^{2k+1} \cos\{(2k+1)\phi\} \right\} \left\{ \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{2\ell+1} \right\}$$

$$= \frac{4V}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b})^{2k+1} \cos\{(2k+1)\phi\}$$

$$= \frac{4V}{\pi} Re \left\{ \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b})^{2k+1} e^{i(2k+1)\phi} \right\}$$

Note the summation

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b} e^{i\phi})^{2k+1} = i \sum_{k=0}^{\infty} \frac{1}{2k+1} (i\frac{\rho}{b} e^{i\phi})^{2k+1} = \frac{1}{2} i \sum_{p=1}^{\infty} \frac{1-(-1)^p}{p} (i\frac{\rho}{b} e^{i\phi})^p$$

$$= \frac{1}{2} i \left\{ \sum_{p=1}^{\infty} \frac{1}{p} (i\frac{\rho}{b} e^{i\phi})^p - \sum_{p=1}^{\infty} \frac{1}{p} (-i\frac{\rho}{b} e^{i\phi})^p \right\}$$

$$= \frac{1}{2} i \ln \left\{ \frac{1 - i\rho e^{i\phi}/b}{1 + i\rho e^{i\phi}/b} \right\}$$

Furthermore

$$\ln \frac{1 - i\rho e^{i\phi}/b}{1 + i\rho e^{i\phi}/b} = \ln \frac{b + \rho\sin\phi - i\rho\cos\phi}{b - \rho\sin\phi + i\rho\cos\phi}$$

$$= \ln \frac{b^2 - \rho^2 - 2ib\rho\cos\phi}{b^2 + \rho^2 - 2b\rho\sin\phi}$$

$$= \ln(Ae^{i\alpha}) = \ln(A) + i\alpha$$

where

$$A = \sqrt{\frac{b^2 + \rho^2 + 2b\rho\sin\phi}{b^2 + \rho^2 - 2b\rho\sin\phi}}, \qquad \alpha = -\tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

Therefore,

$$Re \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (\frac{\rho}{b})^{2k+1} e^{i(2k+1)\phi} \right\} = Re \left\{ \frac{1}{2} i \{\ln(A) + i\alpha\} \right\} = -\frac{1}{2}\alpha = \frac{1}{2} \tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

The potential at $z = L/2$ for $L \gg b$ is

$$\Phi(\rho, \phi) = \frac{2V}{\pi} \tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

agrees with the result of Problem 2.13(a).

Problem 3.14

(a) The potential inside is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r'})G(\vec{r},\vec{r'})d\tau' - \frac{1}{4\pi}\oint \Phi(a,\theta',\phi')\frac{\partial G}{\partial n'}da'$$

Since the sphere is grounded, the potential on the surface $\Phi(a,\theta',\phi') = 0$. The Green function for inside the sphere

$$G(\vec{r},\vec{r'}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

As the result of azimuthal symmetry, $m = 0$, the Green function is simplified:

$$G(\vec{r},\vec{r'}) = \sum_{\ell=0}^{\infty}\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\}P_\ell(\cos\theta)P_\ell(\cos\theta')$$

To proceed further, one needs to figure out the charge density $\rho(r,\theta,\phi)$. Since the density is non-zero only along the $z-$axis and since it is invariant under $z \leftrightarrow -z$, the charge density must of the form:

$$\rho(\vec{r}) \propto \delta(\cos\theta - 1) + \delta(\cos\theta + 1)$$

Furthermore, since the charge density vanishes for $z^2 > d^2$,

$$\rho(\vec{r}) \propto \Theta(d - r)$$

where $\Theta(x)$ is a step function, *i.e.* $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$. Therefore,

$$\rho(r,\theta,\phi) = f(r)\Theta(d-r)\{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\}$$

Since the linear charge density varies as $d^2 - z^2$ along the z and the total charge is Q, one gets the linear charge density along the z as:

$$\rho_z(z) = C(d^2 - z^2), \quad \int_{-d}^{d}\rho_z(z)dz = Q, \quad \Rightarrow \quad C = \frac{3Q}{4d^3}, \quad \rho_z(z) = \frac{3Q}{4d^3}(d^2 - z^2)$$

The charge in a spherical shell of radius $r(< d)$ and thickness dr:

$$2\rho_z(z)dz|_{z=r} = \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\rho(r,\theta,\phi)(r^2dr)$$

The above equation leads to

$$f(r) = \frac{3Q}{8\pi d^3}\frac{d^2 - r^2}{r^2}$$

Therefore, the charge density inside the sphere

$$\rho(\vec{r}) = \frac{3Q}{8\pi d^3}\frac{d^2 - r^2}{r^2}\Theta(d-r)\{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\}$$

The potential inside the sphere

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\rho(\vec{r'})G(\vec{r},\vec{r'})d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{3Q}{8\pi d^3}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta)\cdot 2\pi\cdot\{P_\ell(1) + P_\ell(-1)\}\cdot\int_0^d\frac{d^2 - r'^2}{r'^2}\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right\}r'^2dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3} \sum_{\ell=0}^{\infty} \{1 + (-1)^\ell\} P_\ell(\cos\theta) \int_0^d (d^2 - r'^2) \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right\} dr'$$

For $r > d$:

$$\Phi(\vec{r}) = \frac{3Q}{16\pi\epsilon_0 d^3} \sum_{\ell=0}^{\infty} \{1 + (-1)^\ell\} P_\ell(\cos\theta) \int_0^d (d^2 - r'^2) \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right\} dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3} \sum_{\ell=0}^{\infty} \{1 + (-1)^\ell\} P_\ell(\cos\theta) \int_0^d (d^2 - r'^2) \left\{ \frac{r'^\ell}{r^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right\} dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3} \sum_{\ell=0}^{\infty} \{1 + (-1)^\ell\} P_\ell(\cos\theta) \frac{2d^{\ell+3}}{(\ell+1)(\ell+3)} \left\{ \frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}} \right\}$$

$$= \frac{3Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} P_{2n}(\cos\theta) \frac{d^{2n}}{(2n+1)(2n+3)} \left\{ \frac{1}{r^{2n+1}} - \frac{r^{2n}}{b^{4n+1}} \right\}$$

The integral for the case of $r < d$ is messier and no need to evaluate it.
(b) The surface charge density

$$\sigma = -\epsilon_0 E_\perp|_{r=b} = \epsilon_0 \frac{\partial \Phi}{\partial r}|_{r=b} = -\frac{3Q}{4\pi} \sum_{n=0}^{\infty} \frac{4n+1}{(2n+1)(2n+3)} P_{2n}(\cos\theta) \frac{d^{2n}}{b^{2n+2}}$$

(c) In the limit of $d \to 0$:

$$\Phi(\vec{r}) \Rightarrow \frac{3Q}{4\pi\epsilon_0} P_0(\cos\theta) \frac{1}{3} (\frac{1}{r} - \frac{1}{b}) = \frac{Q}{4\pi\epsilon_0} (\frac{1}{r} - \frac{1}{b})$$

This is the field inside a grounded sphere due to a point charge Q at the origin. The surface charge density

$$\sigma \Rightarrow -\frac{3Q}{4\pi} \frac{1}{3} \frac{1}{b^2} = -\frac{Q}{4\pi b^2}$$

The surface charge is uniformly distributed in this case.

Problem 3.22
The Green function $G(\rho, \phi; \rho', \phi')$ is the solution of the following Poisson equation:

$$\nabla^2 G = -4\pi\delta^3(\vec{r} - \vec{r'}) = -\frac{4\pi}{\rho} \delta(\rho - \rho')\delta(\phi - \phi')$$

From the results of Problem 2.24, for $\phi \neq \phi'$, the angular solution $Q(\phi)$ is of the form $Q_m(\phi) \sim \sin(m\pi\phi/\beta)$ and the functions $\sin(m\pi\phi/\beta)$ are complete:

$$\delta(\phi - \phi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta)$$

Therefore,

$$\nabla^2 G(\rho, \phi; \rho', \phi') = -\frac{8\pi}{\rho\beta} \delta(\rho - \rho') \sum_{m=1}^{\infty} \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta)$$

Expanding the Green function in terms of $\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$:

$$G(\rho, \phi; \rho', \phi') = \sum_{m=1}^{\infty} g_m(\rho, \rho') \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta)$$

and plugging into the above Poisson equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial g_m}{\partial\rho}) - \frac{1}{\rho^2}(\frac{m\pi}{\beta})^2 = -\frac{8\pi}{\beta\rho}\delta(\rho-\rho')$$

For $\rho \neq \rho'$, the above equation reduces to the radial equation of the Poisson equation in Cylindrical coordinates without $z-$dependence, and the two independent solutions are $\rho^{m\pi/\beta}$ and $\rho^{-m\pi/\beta}$:

$$g_m(\rho,\rho') = A_m\rho^{m\pi/\beta} + \frac{B_m}{\rho^{m\pi/\beta}}$$

For $\rho < \rho'$, the boundary condition $g_m(\rho \to 0, \rho') = 0$ leads to

$$B_m = 0, \qquad g_m(\rho,\rho') = A_m(\rho')\rho^{m\pi/\beta}$$

For $\rho > \rho'$, the boundary condition $g_m(\rho = a, \rho') = 0$ leads to

$$A_m = -\frac{B_m}{a^{2m\pi/\beta}}, \qquad g_m(\rho,\rho') = B_m(\rho')\left\{\frac{1}{\rho^{m\pi/\beta}} - \frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

Note that the radial function $g_m(\rho,\rho')$ is invariant under $\rho \leftrightarrow \rho'$, this is only possible if

$$g_m = C_m\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

where $\rho_> = \max(\rho,\rho')$, $\rho_< = \min(\rho,\rho')$ and C_m's are constants independent of ρ and ρ'. Integrating the above radial equation:

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial g_m}{\partial\rho}) - \frac{1}{\rho^2}(\frac{m\pi}{\beta})^2\right\} = -\int_{\rho'-\epsilon}^{\rho'+\epsilon}\frac{8\pi}{\beta\rho}\delta(\rho-\rho')d\rho$$

and letting $\epsilon \to 0$:

$$\frac{\partial g_m}{\partial\rho}|_{\rho'_+} - \frac{\partial g_m}{\partial\rho}|_{\rho'_-} = -\frac{8\pi}{\beta\rho'}$$

Evaluating the above equation:

$$-C_m\frac{1}{\rho'}\frac{m\pi}{\beta}\{1+(\frac{\rho'}{a})^{2m\pi/\beta}\} - C_m\frac{1}{\rho'}\frac{m\pi}{\beta}\{1-(\frac{\rho'}{a})^{2m\pi/\beta}\} = -\frac{8\pi}{\beta\rho'}$$

Therefore, $C_m = 4/m$ and the radial function:

$$g_m(\rho,\rho') = \frac{4}{m}\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

Combining radial and angular solutions, we get the Green function:

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}\sin(\frac{m\pi\phi}{\beta})\sin(\frac{m\pi\phi'}{\beta})$$

Problem 3.24
Useful integral:

$$\int_0^b \rho' J_0(k\rho')d\rho' = \frac{b}{k}J_1(kb); \quad \int_a^b \rho' I_0(k\rho')d\rho' = \frac{b}{k}I_1(kb) - \frac{a}{k}I_1(ka); \quad \int_a^b \rho' K_0(k\rho')d\rho' = \frac{a}{k}K_1(ka) - \frac{b}{k}K_1(kb)$$

(a) Using the results of Problem 3.23, we get three forms of expansions of the Green function ($m = 0$ due to $\phi-$invariance):

$$G(\rho, z; \rho', z') = \frac{4}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)} \sinh(\frac{x_{0n}z_<}{a})\sinh\{\frac{x_{0n}}{a}(L - z_>)\}$$

$$G(\rho, z; \rho', z') = \frac{4}{L}\sum_{n=1}^{\infty} \sin(\frac{n\pi z}{L})\sin(\frac{n\pi z'}{L})\frac{I_0(n\pi\rho_</L)}{I_0(n\pi a/L)}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho_>}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho_>}{L})\right\}$$

$$G(\rho, z; \rho', z') = \frac{8}{La^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(k\pi z/L)\sin(k\pi z'/L)J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

The potential inside the cylinder is given by

$$\Phi(\rho, z) = \frac{1}{4\pi\epsilon_0}\int \rho_c(\vec{r}')G(\vec{r}, \vec{r}')d\tau' - \frac{1}{4\pi}\int \Phi(\vec{r}')\frac{\partial G}{\partial n'}da' = -\frac{V}{2}\int_0^b \frac{\partial G}{\partial z'}|_{z'=L}\rho' d\rho'$$

(i)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{4}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)}\frac{\partial}{\partial z'}\left\{\sinh(\frac{x_{0n}}{a}z)\sinh(\frac{x_{0n}}{a}(L - z'))\right\}|_{z'=L}$$

$$= -\frac{4}{a^2}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{J_1^2(x_{0n})\sinh(x_{0n}L/a)}\sinh(\frac{x_{0n}}{a}z)$$

$$\Phi(\rho, z) = \frac{2V}{a^2}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)\sinh(x_{0n}z/a)}{J_1^2(x_{0n})\sinh(x_{0n}L/a)}\int_0^b \rho' J_0(x_{0n}\rho'/a)d\rho'$$

$$= 2V\frac{b}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_1(x_{0n}b/a)\sinh(x_{0n}z/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)}$$

(ii)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{4\pi}{L^2}\sum_{n=1}^{\infty} n(-1)^n\sin(\frac{n\pi z}{L})\frac{I_0(n\pi\rho_</L)}{I_0(n\pi a/L)}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho_>}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho_>}{L})\right\}$$

For $\rho > b$: $\rho_< = \rho'$ and $\rho_> = \rho$:

$$\Phi(\rho, z) = -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n\sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^b \rho'\frac{I_0(n\pi\rho'/L)}{I_0(n\pi a/L)}d\rho'$$

$$= -2V\frac{b}{L}\sum_{n=1}^{\infty}(-1)^n\sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\frac{I_1(n\pi b/L)}{I_0(n\pi a/L)}$$

For $\rho < b$, the integration has to be break up from $0 \to \rho$ and $\rho \to b$:

$$\Phi(\rho, z) = -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^{\rho}\rho'\frac{I_0(n\pi\rho'/L)}{I_0(n\pi a/L)}d\rho'$$

$$-\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\frac{I_0(n\pi\rho/L)}{I_0(n\pi a/L)}\int_{\rho}^{b}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho'}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho'$$

$$= -\frac{2V}{L}\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n\pi z/L)}{I_0(n\pi a/L)}\left\{I_1(\frac{n\pi\rho}{L})\{\rho I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - \rho K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\}\right.$$

$$\left. + I_0(\frac{n\pi\rho}{L})\{I_0(\frac{n\pi a}{L})(\rho K_1(\frac{n\pi\rho}{L}) - bK_1(\frac{n\pi b}{L})) - K_0(\frac{n\pi a}{L})(bI_1(\frac{n\pi b}{L}) - \rho I_1(\frac{n\pi\rho}{L}))\right\}$$

$$= -\frac{2V}{L}\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n\pi z/L)}{I_0(n\pi a/L)}$$

$$\left\{\rho I_0(\frac{n\pi a}{L})\{I_1(\frac{n\pi\rho}{L})K_0(\frac{n\pi\rho}{L}) - I_0(\frac{n\pi\rho}{L})K_1(\frac{n\pi\rho}{L})\} - bI_0(\frac{n\pi\rho}{L})\{I_0(\frac{n\pi a}{L})K_1(\frac{n\pi b}{L}) + K_0(\frac{n\pi a}{L})I_1(\frac{n\pi b}{L})\}\right\}$$

(iii)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{8\pi}{L^2 a^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k\sin(k\pi z/L)J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

$$\Phi(\rho, z) = -\frac{4\pi V}{L^2 a^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k\sin(k\pi z/L)J_0(x_{0n}\rho/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}\int_0^b \rho' J_0(x_{0n}\rho'/a)d\rho'$$

$$= -4\pi V\frac{b}{L^2 a}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k\sin(k\pi z/L)J_0(x_{0n}\rho/a)J_1(x_{0n}b/a)}{x_{0n}\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

(b) For $L = 4b$ and $a = 2b$:
(i)

$$\Phi(\rho = 0, z = \frac{L}{2}) = V\sum_{n=1}^{\infty}\frac{J_1(x_{0n}/2)\sinh(x_{0n})}{x_{0n}J_1^2(x_{0n})\sinh(2x_{0n})}$$

(ii)

$$\Phi(\rho = 0, z = \frac{L}{2}) = \frac{1}{2}V\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n\pi/2)}{I_0(n\pi/2)}\{I_0(n\pi/2)K_1(n\pi/4) + K_0(n\pi/2)I_1(n\pi/4)\}$$

(iii)

$$\Phi(\rho = 0, z = \frac{L}{2}) = -2\pi V\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k\sin(k\pi/2)J_1(x_{0n}/2)}{x_{0n}\{4x_{0n}^2 + (k\pi)^2\}J_1^2(x_{0n})}$$

No need to work out numerical numbers.