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Due date: October 4, 2001

Physics 505: Solutions to Assignment #4

Problem 3.1

The problem is azimuth symmetric and therefore the general solution is

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}})P_\ell(\cos\theta)$$

The coefficients A_ℓ and B_ℓ are to be determined by the following boundary conditions:

$$\Phi(r=a,\theta) = Vh(\frac{\pi}{2}-\theta); \qquad \Phi(r=b,\theta) = Vh(\theta-\frac{\pi}{2})$$

where $h(x)$ is a step function, *i.e.*, $h(x)=1$ if $x>0$ and $h(x)=0$ if $x<0$. Applying boundary condition at $r=a$:

$$\sum_\ell (A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}})P_\ell(\cos\theta) = Vh(\frac{\pi}{2}-\theta)$$

Multiplying both sides with $P_{\ell'}(\cos\theta)\sin\theta$ and integrating over θ:

$$\sum_\ell (A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}})\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = V\int_0^{\pi/2}P_{\ell'}(\cos\theta)\sin\theta d\theta$$

Note that

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\int_0^{\pi/2}P_{\ell'}(\cos\theta)\sin\theta d\theta = (-\frac{1}{2})^{(\ell'-1)/2}\frac{(\ell'-2)!!}{2(\frac{\ell'+1}{2})!} \qquad (\ell' = \text{odd})$$

$$= \int_0^1 P_{\ell'}(x)dx = \frac{1}{2}\int_{-1}^{+1}P_{\ell'}(x)P_0(x)dx = \delta_{\ell'0} \qquad (\ell' = \text{even})$$

Therefore

$$A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = Vc_\ell$$

$$c_\ell = \begin{cases} \frac{1}{2} & \text{for } \ell = 0 \\ 0 & \text{for other even } \ell \\ (-\frac{1}{2})^{\frac{\ell-1}{2}}\frac{2\ell+1}{4}\frac{(\ell-2)!!}{(\frac{\ell+1}{2})!} & \text{for odd } \ell \end{cases}$$

Applying the boundary condition at $r=b$ leads to a similar equation:

$$A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} = (-1)^\ell Vc_\ell$$

Solving for A_ℓ and B_ℓ:

$$A_\ell = Vc_\ell\frac{a^{\ell+1}-(-1)^\ell b^{\ell+1}}{a^{2\ell+1}-b^{2\ell+1}}, \qquad B_\ell = Vc_\ell(ab)^{\ell+1}\frac{(-1)^\ell a^\ell - b^\ell}{a^{2\ell+1}-b^{2\ell+1}}$$

Therefore

$$A_0 = Vc_0 = \frac{V}{2}; \qquad A_1 = \frac{a^2+b^2}{a^3-b^3}Vc_1 = -\frac{3}{4}V\frac{a^2+b^2}{b^3-a^3}$$

$$A_2 = 0; \qquad A_3 = \frac{a^4+b^4}{a^7-b^7}Vc_3 = \frac{7}{16}V\frac{a^4+b^4}{b^7-a^7}; \qquad A_4 = 0$$

$$B_0 = 0; \qquad B_1 = -(ab)^2\frac{a+b}{a^3-b^3}Vc_1 = \frac{3}{4}V(ab)^2\frac{a+b}{b^3-a^3}; \qquad B_2 = 0$$

$$B_3 = -(ab)^4\frac{a^3+b^3}{a^7-b^7}Vc_3 = -\frac{7}{16}V(ab)^4\frac{a^3+b^3}{b^7-a^7}; \qquad B_4 = 0$$

The potential

$$\Phi(r,\theta) = V\left\{\frac{1}{2} - \frac{3}{4}\{\frac{a^2+b^2}{b^3-a^3}r - (ab)^2\frac{a+b}{b^3-a^3}\frac{1}{r^2}\}P_1(\cos\theta) + \frac{7}{16}\{\frac{a^4+b^4}{b^7-a^7}r^3 - (ab)^4\frac{a^3+b^3}{b^7-a^7}\frac{1}{r^4}\}P_3(\cos\theta) + ...\right\}$$

The limiting case of $a \to 0$:

$$\Phi(r,\theta) \to V\left\{\frac{1}{2} - \frac{3}{4}\frac{r}{b}P_1(\cos\theta) + \frac{7}{16}(\frac{r}{b})^3P_3(\cos\theta) + ...\right\}$$

The limiting case of $b \to \infty$:

$$\Phi(r,\theta) \to V\left\{\frac{1}{2} + \frac{3}{4}(\frac{a}{r})^2P_1(\cos\theta) - \frac{7}{16}(\frac{a}{r})^4P_3(\cos\theta) + ...\right\}$$

Problem 3.4

Using integrals and identities:

$$\int_{-1}^{+1} P_1^1(x)dx = -\frac{\pi}{2}; \qquad \int_{-1}^{+1} P_1^{-1}(x)dx = \frac{\pi}{4}$$

$$\int_{-1}^{+1} P_3^1(x)dx = -\frac{3\pi}{16}; \qquad \int_{-1}^{+1} P_3^{-1}(x)dx = \frac{\pi}{64}$$

$$\int_{-1}^{+1} P_3^3(x)dx = -\frac{45\pi}{8}; \qquad \int_{-1}^{+1} P_3^{-3}(x)dx = \frac{\pi}{128}$$

$$\sin(3\alpha) = -4\sin^3\alpha + 3\sin\alpha$$

This problem is not ϕ symmetric. The general solution is therefore

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}})Y_{\ell m}(\theta,\phi)$$

Since there is no point charge inside the sphere, the potential has to be finite. Therefore, $B_{\ell m} = 0$:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}r^\ell Y_{\ell m}(\theta,\phi)$$

However, the potential is invariant under $\phi \to \phi + 2\pi/n$ transformation, *i.e.*:

$$e^{im\phi} = e^{im(\phi+2\pi/n)}, \quad \Rightarrow \quad \cos(2\frac{m}{n}\pi) = 1 \quad \Rightarrow m = kn, \quad (k = \pm 1, \pm 2, ...)$$

(a) The coefficients $A_{\ell m}$ are to be determined by the potential at the surface:

$$\Phi(r = a, \theta, \phi) = V(-1)^j \ \text{ for } \ \frac{\pi}{n} j \le \phi < \frac{\pi}{n}(j+1) \quad \text{where } j = 0, 1, 2, ..., 2n-1$$

$$A_{\ell m} a^\ell = \int \Phi(r = a, \theta, \phi) Y^*_{\ell m} d\Omega = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \int_{-1}^{1} P^m_\ell(x)dx \int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi$$

Note that

$$\int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi = V \sum_{j=0}^{2n-1} \int_{j\pi/n}^{(j+1)\pi/n} (-1)^j e^{-im\phi} d\phi = \frac{iV}{m}\{e^{-im\pi/n} - 1\} \sum_{j=0}^{2n-1} (-e^{-im\pi/n})^j$$

In order for the above integral to be non-vanishing, m/n must be an odd number. In this case,

$$\sum_{j=0}^{2n-1} (-e^{-im\pi/n})^j = 2n; \qquad \text{and} \qquad \int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi = -4i\frac{Vn}{m}$$

Furthermore, the associated Legendre function $P^m_\ell(x)$ is even if $\ell + m$ =even and odd if $\ell + m$ =odd. Consequently, $A_{\ell m}$'s are non-vanishing only if ℓ and m are either both odd or both even numbers. Therefore,

$$\ell = \text{even} \quad \text{if} \quad n = \text{even}; \qquad \ell = \text{odd} \quad \text{if} \quad n = \text{odd}$$

Since m/n is odd, $m = (2k+1)n, \ k = 0, \pm 1, \pm 2, ...$ The potential inside the sphere is

$$\Phi(r, \theta, \phi) = \sum_{k=-\infty}^{\infty} \sum_{\ell \ge |(2k+1)|n}^{\infty} \frac{1 + (-1)^{\ell+n}}{2} A_{\ell,(2k+1)n} Y_{\ell,(2k+1)n}(\theta, \phi)$$

The first non-vanishing $A_{\ell m}$'s are $A_{n,n}$ and $A_{n,-n}$:

$$A_{n,n} = \frac{-4iV}{a^n} \sqrt{\frac{2n+1}{4\pi}\frac{1}{(2n)!}} \int_{-1}^{+1} P^n_n(x)dx; \quad A_{n,-n} = \frac{4iV}{a^n} \sqrt{\frac{2n+1}{4\pi}(2n)!} \int_{-1}^{+1} P^{-n}_n(x)dx$$

(b) For $n = 1$, the non-vanishing $A_{\ell m}$'s up to $\ell = 3$ are $A_{11}, A_{1,-1}, A_{31}, A_{3,-1}, A_{33}, A_{3,-3}$:

$$A_{11} = \frac{-4iV}{a} \sqrt{\frac{3}{8\pi}} \int_{-1}^{+1} P^1_1(x)dx = \frac{iV}{a}\sqrt{\frac{3\pi}{2}}; \qquad A_{1,-1} = \frac{4iV}{a} \sqrt{\frac{3}{2\pi}} \int_{-1}^{+1} P^{-1}_1(x)dx = \frac{iV}{a}\sqrt{\frac{3\pi}{2}} = A_{11}$$

$$A_{31} = \frac{-4iV}{a^3} \sqrt{\frac{7}{48\pi}} \int_{-1}^{+1} P^1_3(x)dx = \frac{iV}{16a^3}\sqrt{21\pi}; \qquad A_{3,-1} = \frac{4iV}{a^3} \sqrt{\frac{84}{4\pi}} \int_{-1}^{+1} P^{-1}_3(x)dx = \frac{iV}{16a^3}\sqrt{21\pi} = A_{31}$$

$$A_{33} = -\frac{4iV}{3a^3} \sqrt{\frac{7}{2880\pi}} \int_{-1}^{+1} P^3_3(x)dx = \frac{iV}{16a^3}\sqrt{35\pi}; \qquad A_{3,-3} = \frac{4iV}{3a^3} \sqrt{\frac{1260}{\pi}} \int_{-1}^{+1} P^{-3}_3(x)dx = \frac{iV}{16a^3}\sqrt{35\pi} = A_{33}$$

The potential expansion (up to the term $\ell = 3$) becomes:

$$\Phi(r, \theta, \phi) = r(A_{11}Y_{11} + A_{1,-1}Y_{1,-1}) + r^3(A_{31}Y_{31} + A_{3-1}Y_{3,-1} + A_{33}Y_{33} + A_{3-3}Y_{3,-3} + ...$$

$$= rA_{11}(Y_{11} - Y^*_{11}) + r^3 A_{31}(Y_{31} - Y^*_{31}) + r^3 A_{33}(Y_{33} - Y^*_{33})$$

where

$$rA_{11}(Y_{11}-Y_{11}^*) = 2iV\sqrt{\frac{3\pi}{2}}\frac{r}{a}\sqrt{\frac{3}{8\pi}}\sin\theta(e^{-i\phi}-e^{i\phi}) = \frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi$$

$$r^3A_{31}(Y_{31}-Y_{31}^*) = \frac{iV}{8}\sqrt{21\pi}\frac{r^3}{a^3}\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta-1)(e^{-i\phi}-e^{i\phi}) = \frac{21}{64}V\frac{r^3}{a^3}\sin\theta(5\cos^2\theta-1)\sin\phi$$

$$r^3A_{33}(Y_{33}-Y^*33) = \frac{iV}{8}\sqrt{35\pi}\frac{r^3}{a^3}\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta(e^{-i3\phi}-e^{i3\phi}) = \frac{35}{64}V\frac{r^3}{a^3}\sin^3\theta\sin(3\phi)$$

Combining the above three terms, we have

$$\Phi(r,\theta,\phi) = V\left\{\frac{3}{2}(\frac{r}{a})\sin\theta\sin\phi + \frac{7}{64}(\frac{r}{a})^3\{3\sin\theta(5\cos^2\theta-1)\sin\phi+5\sin^3\theta\sin(3\phi)\}\right\}+...$$

$$= V\left\{\frac{3}{2}(\frac{r}{a})\sin\theta\sin\phi + \frac{7}{64}(\frac{r}{a})^3\{3\sin\theta(5\cos^2\theta-1)-20\sin^3\theta\sin^3\phi+15\sin^3\theta\sin\phi\}\right\}+...$$

Translating to Cartesian coordinates ($x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, and $z=r\cos\theta$):

$$\Phi(x,y,z) = V\left\{\frac{3}{2}\frac{y}{a}+\frac{7}{64}\{3\frac{y}{a}(5\frac{z^2}{a^2}-\frac{r^2}{a^2})-20\frac{y^3}{a^3}+15\frac{y}{a}(\frac{r^2}{a^2}-\frac{z^2}{a^5})\}\right\}$$

Rotating coordinates ($x\to y$, $y\to z$, and $z\to x$):

$$\Phi(x,y,z) = V\left\{\frac{3}{2}\frac{z}{a}+\frac{7}{64}\{3\frac{z}{a}(5\frac{x^2}{a^2}-\frac{r^2}{a^2}-20\frac{z^3}{a^3}+15\frac{z}{a}(\frac{r^2}{a^2}-\frac{x^2}{a^2})\}\right\}$$

Translating back to Spherical coordinates in the rotated system ($x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, and $z=r\cos\theta$):

$$\Phi(r,\theta,\phi) = V\left\{\frac{3}{2}\frac{r}{a}\cos\theta+\frac{7}{64}\frac{r^3}{a^3}\{3\cos\theta(5\sin^2\theta\cos^2\phi-1)-20\cos^3\theta+15\cos\theta(1-\sin^2\theta\sin^2\phi)\}\right\}$$

$$= V\left\{\frac{3}{2}\frac{r}{a}\cos\theta-\frac{7}{8}\frac{r^3}{a^3}\{\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\}\right\}$$

$$= V\left\{\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{8}\frac{r^3}{a^3}P_3(\cos\theta)+...\right\}$$

agrees with Eq. (3.36).

Problem 3.7

(a) The problem is $\phi-$symmetric. The potential for an arbitrary point (r,θ,ϕ) is

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r_+}+\frac{1}{r_-}-\frac{2}{r}\right\}$$

where r_+ and r_- are distances from the point to the charge at $z=+a$ and $z=-a$ respectively:

$$r_+ = \sqrt{r^2+a^2-2ar\cos\theta};\qquad r_- = \sqrt{r^2+a^2+2ar\cos\theta}$$

Therefore,

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}+\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}}-\frac{2}{r}\right\}$$

To find the limiting form of the potential as $a \to 0$, we expand r_+ and r_- as:

$$\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1-2\frac{a}{r}\cos\theta+(\frac{a}{r})^2}} = \sum_{\ell=0}^{\infty}\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

$$\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1+2\frac{a}{r}\cos\theta+(\frac{a}{r})^2}} = \sum_{\ell=0}^{\infty}(-1)^\ell\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

Using these expansions, the potential can be written as

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta) - \frac{2}{r}\right\} = \frac{q}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{a^{2n}}{r^{2n+1}}P_{2n}(\cos\theta)$$

In the limit of $a \to 0$ while keeping qa^2 constant:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{2\frac{a^2}{r^3}P_2(\cos\theta)+2\frac{a^4}{r^5}P_4(\cos\theta)+....\right\} \to \frac{Q}{2\pi\epsilon_0}\frac{1}{r^3}P_2(\cos\theta)$$

(b) With the grounded sphere, the potential at (r,θ,ϕ) are superpositions of those of the three charges and their imagine charges. Denoting Φ_+, Φ_- and Φ_0 the potentials of charges at $z=+a,-a,0$ and of their respective imagine charges, we have

$$\Phi_+(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{-qb/a}{\sqrt{r^2+(b^2/a)^2-2r(b^2/a)\cos\theta}}\right\}$$

$$\Phi_-(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} + \frac{-qb/a}{\sqrt{r^2+(b^2/a)^2+2r(b^2/a)\cos\theta}}\right\}$$

$$\Phi_0(r,\theta) = \frac{1}{4\pi\epsilon_0}\lim_{r_0\to 0}\left\{\frac{-2q}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} + \frac{2qb/r_0}{\sqrt{r^2+(b^2/r_0)^2-2r(b^2/r_0)\cos\theta}}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{-\frac{1}{r}+\lim_{r_0\to 0}\frac{1}{b\sqrt{1+(rr_0/b)^2-2(rr_0/b^2)\cos\theta}}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{-\frac{1}{r}+\frac{1}{b}\lim_{r_0\to 0}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta)(\frac{rr_0}{b^2})^\ell\right\} = \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}\right\}$$

For $r < a$, Φ_+ and Φ_- can be expanded in terms of r/a or ar/b:

$$\Phi_+ = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{a}\frac{1}{\sqrt{1+(\frac{r}{a})^2-2(\frac{r}{a})\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2-2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

$$\Phi_- = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{a}\frac{1}{\sqrt{1+(\frac{r}{a}^2+2\frac{r}{a}\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2+2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}(-1)^\ell\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

Adding Φ_+, Φ_- and Φ_0 together:

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\{(1+(-1)^\ell\}\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)+\frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b} - \frac{1}{r} + \sum_{n=0}^{\infty} r^{2n}\{\frac{1}{a^{2n+1}} - \frac{a^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

For $r > a$, Φ_+ and Φ_- have to be expanded in terms of a/r or ar/b:

$$\Phi_+ = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r}\frac{1}{\sqrt{1+(\frac{a}{r})^2 - 2\frac{a}{r}\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2 - 2\frac{ar}{b^2}\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left\{\frac{a^\ell}{r^{\ell+1}} - \frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

$$\Phi_- = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r}\frac{1}{\sqrt{1+(\frac{a}{r})^2 + 2\frac{a}{r}\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2 + 2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}(-1)^\ell\left\{\frac{a^\ell}{r^{\ell+1}} - \frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

The total potential for $r > a$:

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\{(1+(-1)^\ell\}\left\{\frac{a^\ell}{r^{\ell+1}} - \frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta) + \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b} - \frac{1}{r}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b} - \frac{1}{r} + \sum_{n=0}^{\infty}\{\frac{a^{2n}}{r^{2n+1}} - \frac{a^{2n}r^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\sum_{n=1}^{\infty} a^{2n}\left\{\frac{1}{r^{2n+1}} - \frac{r^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

In the limit of $a \to 0$:

$$\Phi(r,\theta) \to \frac{1}{2\pi\epsilon_0}qa^2\left\{\frac{1}{r^3} - \frac{r^2}{b^5}\right\}P_2(\cos\theta) = \frac{1}{2\pi\epsilon_0}\frac{Q}{r^3}\left\{1 - \frac{r^5}{b^5}\right\}P_2(\cos\theta)$$

Problem 3.9

Useful integrals:

$$\int_0^{2\pi}\sin(m\phi)\sin(n\phi)d\phi = \pi\delta_{mn}, \qquad \int_0^{2\pi}\cos(m\phi)\cos(n\phi)d\phi = \pi\delta_{mn}$$

$$\int_0^{L}\sin(m\pi z/L)\sin(n\pi z/L)dz = \frac{L}{2}\delta_{mn}, \qquad \int_0^{L}\cos(m\pi z/L)\cos(n\imath z/L)dz = \frac{L}{2}\delta_{mn}$$

This is a problem of solving Laplace's equation $\nabla^2\Phi = 0$ with the following boundary conditions:

$$\Phi(\rho,\phi,z=0) = 0, \quad \Phi(\rho,\phi,z=L) = 0, \quad \Phi(\rho=b,\phi,z) = V(\phi,z)$$

Assuming the solution has the form:

$$\Phi(\rho,\phi,z) \sim R(\rho)Q(\phi)Z(z)$$

and plugging into the Laplace's equation, one gets:

$$Q(\phi) \sim A\cos(\nu\phi) + B\sin(\nu\phi), \quad Z(z) \sim C\cos(kz) + D\sin(kz), \quad R(\rho) \sim EI_\nu(k\rho) + FK_\nu(k\rho)$$

where A, B, C, D, E, F are constants. Applying generation consideration and boundary conditions:

- $Q(\phi + 2\pi) = Q(\phi) \quad \Rightarrow \quad \nu = m,\ m = 0, 1, 2, ...$
- Finite potential at $\rho = 0 \quad \Rightarrow \quad F = 0$;

- $\Phi = 0$ at $z = 0 \quad \Rightarrow \quad C = 0$;
- $\Phi = 0$ at $z = L \quad \Rightarrow \quad kL = n\pi, \ n = 0, 1, 2, ...$

Therefore, the complete general solution of the potential is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}\rho) \sin(\frac{n\pi}{L}z)\{A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\}$$

The coefficients A_{mn} and B_{mn} are to be determined from the boundary condition at $\rho = b$:

$$V(\phi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z)\{A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\}$$

This is a double Fourier transform of $V(\phi, z)$. Multiplying $\sin(m'\phi)$ and integrating over ϕ:

$$\int_0^{2\pi} d\phi V(\phi, z)\sin(m'\phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z) A_{mn} \cdot (\pi\delta_{m'm}) = \pi \sum_{n=0}^{\infty} I_{m'}(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z) A_{m'n}$$

Multiplying $\sin(\frac{n'\pi}{L}z)$ and integrating over z:

$$\int_0^L dz \int_0^{2\pi} d\phi V(\phi, z)\sin(m'\phi)\sin(\frac{n'\pi}{L}z) = \pi \sum_{n=0}^{\infty} I'_m(\frac{n\pi}{L}b) A_{m'n} \cdot (\frac{L}{2}\delta_{n'n}) = \pi\frac{L}{2} I_{m'}(\frac{n'\pi}{L}b) A_{m'n'}$$

Therefore

$$A_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} = \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z)\sin(m\phi)\sin(\frac{n\pi}{L}z)$$

Similarly

$$B_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} = \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z)\cos(m\phi)\sin(\frac{n\pi}{L}z)$$

As usual, B_{0n} is to be replaced by $B_{0n}/2$.