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Physics 505: Solutions to Assignment #3

Problem 2.10

The problem is very similar to a grounded conducting sphere in a uniform electric field E_0, the case discussed in Section 2.5. The region between the two plates is identical to one of the half spaces of the above mentioned problem, where the half spaces are defined by a plane perpendicular to the field and cut the sphere into two hemispheres.
(a) In a spherical coordinate system centered at the origin of the boss with $+z$ pointing to the other conductor, the potential between the two plates is given by Eq. (2.14):

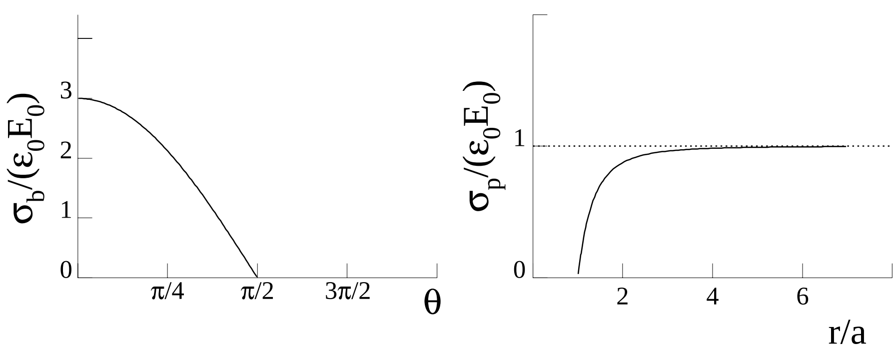
$$\Phi(r,\theta) = -E_0(r - \frac{a^3}{r^2})\cos\theta$$

The surface charge density on the boss:

$$\sigma_b(a,\theta) = \epsilon_0 E_r|_{r=a} = -\epsilon_0 \frac{\partial\Phi}{\partial r}|_{r=a} = 3\epsilon_0 E_0 \cos\theta \qquad 0 < \theta < \frac{\pi}{2}$$

The density on the plane:

$$\sigma_p(r,\theta=\frac{\pi}{2}) = \epsilon_0 E_\perp|_{\theta=\pi/2} = -\epsilon_0 E_\theta|_{\theta=\pi/2} = \epsilon_0 \frac{1}{r}\frac{\partial\Phi}{\partial\theta}|_{\theta=\pi/2} = \epsilon_0 E_0(1-\frac{a^3}{r^3})$$



(b) The total charge on the boss

$$Q_b = \int \sigma_b da = 3\epsilon_0 E_0 a^2 \int_0^{\pi/2} \cos\theta\sin\theta d\theta \int_0^{2\pi} d\phi = 3\pi\epsilon_0 E_0 a^2$$

(c) The boundary condition $\Phi = 0$ on the plane and the boss can be meet by placing three image charges, one $-qa/d$ at $z = a^2/d$, one $+qa/d$ at $z = -a^2/d$ and the third one $-q$ at $z = -d$. The induced charge distribution on the boss is the sum of those induced by charge pairs $(q, -qa/d)$ and $(-q, qa/d)$:

$$\sigma(\theta) = \sigma_1(\theta) + \sigma_2(\theta)$$

where σ_1 and σ_2 are given by Eq. (2.5):

$$\sigma_1(\theta) = -\frac{q}{4\pi a^2}\frac{a(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{where} \quad 0 < \theta < \frac{\pi}{2} \quad \text{due to charges} \quad (q, -qa/d)$$

$$\sigma_2(\theta) = +\frac{q}{4\pi a^2}\frac{a(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{where} \quad \frac{\pi}{2} < \theta < \pi \quad \text{due to charges} \quad (-q, qa/d)$$

The total induced charge on the boss

$$q' = \int(\sigma_1+\sigma_2)da = -\frac{q}{4\pi a^2}a(d^2-a^2)2\pi a^2\{\int_0^{\pi/2} d\theta - \int_{\pi/2}^{\pi} d\theta\}\frac{\sin\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} = -q(1-\frac{d^2-a^2}{d\sqrt{d^2+a^2}})$$

Problem 2.11

The electric field and potential due to a line charge are

$$\vec{E}(\rho,\phi) = \frac{\tau}{2\pi\epsilon_0}\frac{1}{\rho}\frac{\vec{\rho}}{\rho}, \quad \Phi(\rho,\phi) = C - \frac{\tau}{2\pi\epsilon_0}\ln\rho$$

where C, τ and ρ are, respectively, a constant, the line charge density, and the distance to the line charge. Note that the potential does not depend on z (along the axis).

(a) The image charge has to be in the plane formed by the cylinder axis and the line charge. Choose a cylindrical coordinate system with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis. In this case, the line charge is at $x = R$. Let τ' and R' be the image line charge density and the polar distance to the axis, the potential at a point (ρ, ϕ) is given by

$$\Phi(\rho,\phi) = \Phi_0 - \frac{\tau}{2\pi\epsilon_0}\ln r - \frac{\tau'}{2\pi\epsilon_0}\ln r'$$

Φ_0 is another constant. r and r' are distances from the point to the line charge τ and the image charge τ':

$$r = \sqrt{\rho^2+R^2-2\rho R\cos\phi}; \qquad r' = \sqrt{\rho^2+R'^2-2\rho R'\cos\phi}$$

Therefore

$$\Phi(\rho,\phi) = \Phi_0 - \frac{1}{4\pi\epsilon_0}\{\tau\ln(\rho^2+R^2-2R\rho\cos\phi) + \tau'\ln(\rho^2+R'^2-2R'\rho\cos\phi)\}$$

At the surface of the cylinder ($\rho = b$):

$$\Phi(b,\phi) = \Phi_0 - \frac{1}{4\pi\epsilon_0}\{\tau\ln(b^2+R^2-2Rb\cos\phi) + \tau'\ln(b^2+R'^2-2R'b\cos\phi)\} = \text{Constant}$$

independent of ϕ. This is only possible if

$$\begin{aligned} \tau' &= -\tau \\ b^2+R^2 &= A(b^2+R'^2) \\ bR &= A(bR') \end{aligned}$$

A is another constant. It is easy to determine R' from the above equation to be $R' = b^2/R$. Therefore, the line image charge is at $x = +b^2/R$ with a line charge density $-\tau$.

(b) Potential at any point (ρ, ϕ):

$$\Phi(\rho,\phi) = \Phi_0 - \frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho^2+R^2-2R\rho\cos\phi}{\rho^2+(b^2/R)^2-2(b^2/R)\rho\cos\phi}\}$$

As $\rho\to\infty$, $\Phi(\rho,\phi)\to\Phi_0$. Since the potential vanishes at infinity, one gets Φ_0. The potential at a point (ρ, ϕ):

$$\Phi(\rho,\phi) = -\frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho^2+R^2-2R\rho\cos\phi}{\rho^2+(b^2/R)^2-2(b^2/R)\rho\cos\phi}\}$$

Note that the potential on the cylinder surface is

$$\Phi(b,\phi) = -\frac{\tau}{2\pi\epsilon_0}\ln\{\frac{R}{b}\}$$

For a point far away from the cylinder,

$$\frac{\rho^2 + R^2 - 2R\rho\cos\phi}{\rho^2 + (b^2/R)^2 - 2(b^2/R)\rho\cos\phi} \approx \frac{1 - 2(R/\rho)\cos\phi}{1 - 2(b^2/R\rho)\cos\phi} \approx 1 + 2\frac{b^2 - R^2}{R\rho}\cos\phi$$

Therefore the asymptotic form of the potential far from the cylinder:

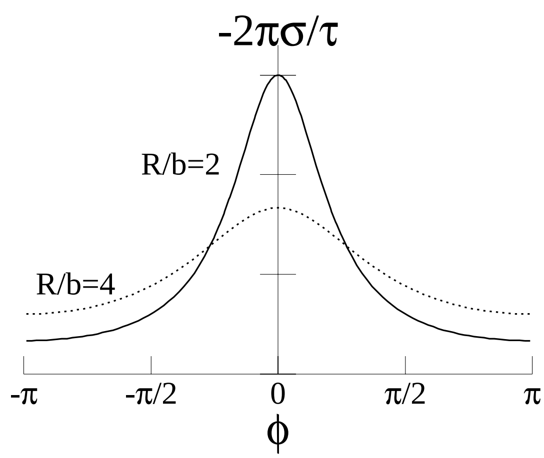
$$\Phi(\rho, \phi) \approx -\frac{\tau}{4\pi\epsilon_0}\ln\{1 + 2\frac{b^2 - R^2}{R\rho}\cos\phi\} \approx -\frac{\tau}{2\pi\epsilon_0}\frac{b^2 - R^2}{R\rho}\cos\phi$$

(c) Since the cylinder has a constant potential, the potential inside the cylinder must be the same constant (see the next problem for proof). Therefore, the surface charge density on the cylinder

$$\sigma(\phi) = \epsilon_0 E_\rho|_{\rho=b} = -\epsilon_0\frac{\partial\Phi}{\partial\rho}|_{\rho=b} = -\frac{\tau}{2\pi}\frac{R^2 - b^2}{b(b^2 + R^2 - 2bR\cos\phi)}$$

Note that the total surface charge per unit length

$$\tau'' = \int_0^{2\pi}\sigma(\phi)bd\phi = -\tau$$



(d) The force on the charge per unit length:

$$\vec{F} = \tau\vec{E} = -\frac{\tau^2}{2\pi\epsilon_0}\frac{1}{R - b^2/R}\frac{\vec{\rho}}{\rho} = -\frac{\tau^2}{2\pi\epsilon_0}\frac{R}{R^2 - b^2}\frac{\vec{\rho}}{\rho}$$

where $\vec{E}$ is the electric field at the line charge due to the image charge. The force is attractive.

The Green function for the Dirichlet condition

It is interesting to derive the Green function for the Dirichlet condition for the case of cylinder. The Dirichlet condition $\Phi(b, \phi) = 0$ leads to $\Phi_0 = \frac{\tau}{4\pi\epsilon_0}\ln(R^2/b^2)$. The potential at (ρ, ϕ) due to the line charge τ at (ρ', ϕ') with the Dirichlet boundary condition is:

$$\Phi(\rho, \phi) = \frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho'^2[\rho^2 + (b^2/\rho')^2 - 2(b^2/\rho')\rho\cos(\phi' - \phi)]}{b^2[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi' - \phi)]}\}$$

The Green function is the potential at (ρ, ϕ) due to a line charge with density $\tau = 4\pi\epsilon_0$ at (ρ', ϕ'):

$$G_D(\rho, \phi, \rho', \phi') = \ln\{\frac{\rho^2\rho'^2 + b^4 - 2b^2\rho\rho'\cos(\phi' - \phi)}{b^2[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi' - \phi)]}\}$$

Problem 2.13

Integrals and identities useful for the problem:

$$\int_0^{2\pi} \frac{d\phi}{a^2+b^2-2ab\cos\phi} = \frac{2\pi}{|a-b|(a+b)}$$

$$\int \frac{\cos\phi d\phi}{a^2-b^2\cos^2\phi} = \frac{\tan^{-1}(b\sin\phi/\sqrt{a^2-b^2})}{b\sqrt{a^2-b^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

(a) The cylinder can be viewed as a superposition of two cylinders with the following potentials: We can regard this problem as a superposition of two problems:

$$\Phi_1(b,\phi) = (V_1+V_2)/2$$

$$\Phi_2(b,-\frac{\pi}{2}<\phi<\frac{\pi}{2}) = \frac{V_1-V_2}{2} \qquad \Phi_2(b,\frac{\pi}{2}<\phi<\frac{3\pi}{2}) = \frac{V_2-V_1}{2}$$

The potential inside the first cylinder (see problem 2.12):

$$\Phi_1(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{V_1+V_2}{2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' = \frac{V_1+V_2}{2}$$

The potential inside the second cylinder:

$$\Phi_2(\rho,\phi) = \Phi_{2a}(\rho,\phi)+\Phi_{2b}(\rho,\phi)$$

where

$$\Phi_{2a}(\rho,\phi) = \frac{V_1-V_2}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

$$\Phi_{2b}(\rho,\phi) = -\frac{V_1-V_2}{4\pi}\int_{\pi/2}^{3\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

Let $\phi''=\phi'-\pi$, then

$$\Phi_{2b}(\rho,\phi) = -\frac{V_1-V_2}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2+2b\rho\cos(\phi''-\phi)}d\phi''$$

Combining Φ_{2a} and Φ_{2b}:

$$\Phi_2(\rho,\phi) = \frac{V_1-V_2}{4\pi}(b^2-\rho^2)\int_{-\pi/2}^{\pi/2}\frac{4b\rho\cos(\phi'-\phi)}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}d\phi'$$

$$= \frac{(V_1-V_2)}{\pi}b\rho(b^2-\rho^2)\frac{1}{2b\rho(b^2-\rho^2)}\tan^{-1}\{\frac{2b\rho}{b^2-\rho^2}\sin(\phi'-\phi)\}|_{\phi'=-\pi/2}^{\phi'=\pi/2}$$

$$= \frac{V_1-V_2}{\pi}\tan^{-1}(\frac{2b\rho}{b^2-\rho^2}\cos\phi)$$

Thus the potential inside the original cylinder

$$\Phi(\rho,\phi) = \Phi_1(\rho,\phi) + \Phi_2(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\{\frac{2b\rho}{b^2-\rho^2}\cos\phi\}$$

(b) Let σ be the surface charge density, the electric fields just inside and outside the surface are given by Gauss's law:

$$\vec{E}_{\rm in} = -\frac{\sigma}{2\epsilon_0}\frac{\vec{\rho}}{\rho}, \qquad \vec{E}_{\rm out} = \frac{\sigma}{2\epsilon_0}\frac{\vec{\rho}}{\rho}$$

Therefore,

$$\sigma = -2\epsilon_0\vec{E}_{\rm in}\cdot\frac{\vec{\rho}}{\rho}|_{\rho=b} = -2\epsilon_0 E_\rho|_{\rho=b} = 2\epsilon_0\frac{\partial\Phi}{\partial\rho}|_{\rho=b} = \frac{2\epsilon_0(V_1-V_2)}{\pi b\cos\phi}$$

Problem 2.21

First of all, the Poisson integral solution is the solution of Problem 2.12, *i.e.*, the potential inside a cylinder:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

Cauchy's theorem is expressed using complex variable z where $z=\rho e^{i\phi}$. Let z be inside the curve C and $F(z)=\Phi(z)$, we then have

$$\Phi(z) = \frac{1}{2\pi i}\oint\frac{\Phi(z')}{z'-z}dz'$$

Exploiting the hint, the image point of z is $(b^2/|z|)e^{i\phi} = b^2/z^*$ (here z^* is the complex conjugate of z) which lies outside the curve C. Therefore

$$\frac{1}{2\pi i}\oint\frac{\Phi(z')}{z'-b^2/z^*}dz' = 0$$

Subtract this zero integral from the $\Phi(z)$:

$$\Phi(z) = \frac{1}{2\pi i}\oint\Phi(z')\{\frac{1}{z'-z} - \frac{1}{z'-b^2/z^*}\}dz'$$

For the Poisson integral problem, the curve C is circle of radius b, z' is on the curve, *i.e.*, $z'z'^* = b^2$. Using the identity

$$z' - \frac{b^2}{z^*} = z' - z'z'^*/z^* = \frac{z'}{z^*}(z^*-z'^*)$$

one gets

$$\frac{1}{z'-z} - \frac{1}{z'-b^2/z^*} = \frac{1}{z'-z} + \frac{z^*/z'}{z'^*-z^*} = \frac{z'^*-z^*+z^*-zz^*/z'}{|z'-z|^2} = \frac{1}{z'}\frac{z'z^*-zz^*}{|z'-z|^2}$$

Plugging in to the potential $\Phi(z)$:

$$\Phi(z) = \frac{1}{2\pi i}\oint\Phi(z')z'\frac{z'z'^*-zz^*}{|z'-z|^2}\frac{dz'}{z'}$$

In polar coordinate system, $z=\rho e^{i\phi}$, $z'=be^{i\phi'}$, therefore

$$zz^* = \rho^2, \qquad z'z'^* = b^2, \qquad \frac{dz'}{z'} = id\phi'$$

$$|z'-z|^2 = \rho^2 + b^2 - 2b\rho\cos(\phi'-\phi)$$

The potential inside the curve in polar coordinates

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$