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Due date: September 20, 2001

Physics 505: Solutions to Assignment #2

Problem 2.2
(a) Let the point charge q at $\vec{r}_o$ and the image charge q_i at $\vec{r}_i$. The potential for a point $\vec{r}$ is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_o|} + \frac{1}{4\pi\epsilon_0}\frac{q_i}{|\vec{r}-\vec{r}_i|}$$

On the surface of the sphere, the potential is zero every where, *i.e.*, $\Phi(r=a)=0$. This is only possible if

$$q_i = -q\frac{a}{r_o} \qquad \text{and} \qquad r_i = \frac{a^2}{r_o}.$$

The potential inside the sphere is therefore

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_o|} - \frac{1}{4\pi\epsilon_0}\frac{a}{r_o}\frac{q}{|\vec{r}-a^2\vec{r}_o/r_o^2|}$$

or

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\{\frac{1}{\sqrt{r^2+r_o^2-2rr_o\cos\beta}} - \frac{a}{\sqrt{r^2r_o^2+a^4-2a^2rr_o\cos\beta}}\}$$

where β is the angle between $\vec{r}$ and $\vec{r}_o$.
(b) The induced inner surface charge density

$$\sigma = \epsilon_0\frac{\partial\Phi}{\partial r}|_{r=a} = \frac{q}{4\pi a^2}\frac{a}{r_o}\frac{1-\frac{a^2}{r_o^2}}{(1+\frac{a^2}{r_o^2}-2\frac{a}{r_o}\cos\beta)^{3/2}} = -\frac{q}{4\pi a^2}\frac{a(a^2-r_o^2)}{(a^2+r_o^2-2ar_o\cos\beta)^{3/2}}$$

Note the total charge on the inner surface is

$$\oint \sigma da = -q \qquad (a > r_o)$$

There is no charge on the outer surface.
(c) The force on charge q

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\cdot q_i}{(r_o-r_i)^2}\frac{\vec{r}_o-\vec{r}_i}{|\vec{r}_o-\vec{r}_i|} = \frac{q^2}{4\pi\epsilon_0}\frac{ar_o}{(a^2-r_o^2)^2}\frac{\vec{r}_o}{r_o}$$

(d) If the sphere is kept at a fixed potential V:

(a) The potential inside is raised by a constant V

(b) No change to the inner surface charge density, but there will be uniformly distributed charge on the outer surface.

(c) No change to the force.

If the sphere has a total charge Q:

(a) The potential inside is raised by a constant equal to the potential of the sphere.

(b) No change to the inner surface charge density, but there will be $Q+q$ uniformly distributed on the outer surface.

(c) No change to the force.

Problem 2.4
(a) The force on the point charge q is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]\frac{\vec{d}}{d}$$

At the point where $F = 0$, the force changes from repulsive to attractive. Solving

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}],$$

one gets (with the help of the `Mathematica`)

$$d = \frac{1}{2}(1 + \sqrt{5})R$$

Therefore at a distance smaller than $d - R = \frac{1}{2}(\sqrt{5} - 1)R \approx 0.618R$ from the surface, the charge q is attracted to the sphere.
(b) When the charge is close to the surface $d = R + a \sim R$. The force

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}] \approx \frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}[1 - \frac{R^5}{R((R+a)^2 - R^2)^2}] \approx -\frac{1}{16\pi\epsilon_0}\frac{q^2}{a^2}$$

(c) The limiting force on the point charge q is determined by itself and its image charge and therefore is independent of the charge on the conductor. However, the location where the force changes between repulsive and attractive is a function of the charge on the sphere.
For $Q = 2q$,

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[2 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]$$

Using for example the `Mathematica` program to solve the above equation, one gets $d = 1.4276R$, *i.e.*, the switch point is a distance $0.4276R$ away from the surface.
For $Q = \frac{1}{2}q$,

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1/2 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]$$

Again, with the help of programs like `Mathematica`, one gets $d = 1.8823R$. The switch over is a distance $0.8823R$ away from the surface.

Problem 2.5
(a) The electric force on the charge q is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\frac{(a/r)^3}{(1 - a^2/r^2)^2}\frac{\vec{r}}{r}$$

The work done to the charge

$$W = -\int_r^\infty \vec{F}\cdot d\vec{\ell} = \int_r^\infty |F(r')|dr' = \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{dr'}{r'^3(1 - a^2/r'^2)^2} = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}$$

Alternatively, we could consider calculating W from the energy conservation:

$$U_q + U_{q'} + U_{qq'} + W = U_q \quad \Rightarrow \quad W = -U_{q'} - U_{qq'}$$

where U_q and $U_{q'}$ are self energies of the point charge q and the induced charge on the sphere and $U_{qq'}$ is the interaction energy between q and the induced charge at r. The problem with this approach is that $U_{q'}$ is difficult to calculate.

(b) The work done to the charge

$$W = -\int_r^\infty \vec{F}\cdot d\vec{\ell} = -\frac{q}{4\pi\epsilon_0}[\int_r^\infty \frac{Q}{r'^2}dr' - qa^3\int_r^\infty \frac{2r'^2-a^2}{r'(r'^2-a^2)^2}dr'] = \frac{q^2 a}{8\pi\epsilon_0(r^2-a^2)} - \frac{1}{4\pi\epsilon_0}[\frac{q^2 a}{2r^2}+\frac{qQ}{r}]$$

On the approach of energy conservation, see the discussion above.

Problem 2.9

(a) The surface charge density on the sphere is (Eq. 2.15)

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

The electrostatic pressure

$$\vec{P} = \frac{\sigma^2}{2\epsilon_0}\vec{n}$$

The total electrostatic force on one of the hemisphere

$$F = \int P_z da = a^2 \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi \frac{\sigma^2}{2\epsilon_0}\cos\theta = \frac{9}{4}\pi a^2 \epsilon_0 E_0^2$$

(b) The surface charge density on the sphere is

$$\sigma = 3\epsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2}$$

The total force

$$F = \int P_z da = a^2 \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi \frac{\sigma^2}{2\epsilon_0}\cos\theta = \frac{9}{4}\pi a^2 \epsilon_0 E_0^2 + \frac{1}{2}E_0 Q + \frac{1}{32\pi\epsilon_0}\frac{Q^2}{a^2}$$