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Physics 505: Solutions to Assignment #1

Problem 1.5

From Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$, we have the charge density $\rho = -\epsilon_0\nabla^2\Phi$. Let $f \equiv \frac{1}{r}$ and $g \equiv e^{-\alpha r}(1+\frac{\alpha r}{2})$, then

$$\rho = -\epsilon_0\nabla^2\Phi = -\frac{q}{4\pi}(g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g)$$

Note that

$$\nabla f = -\frac{\vec{r}}{r^3}, \quad \nabla^2 f = -4\pi\delta^3(\vec{r})$$

$$\nabla g = -\frac{1}{2}\alpha e^{-\alpha r}(1+\alpha r)\frac{\vec{r}}{r}, \quad \nabla^2 g = \frac{\alpha}{2r}e^{-\alpha r}(-2-2\alpha r+\alpha^2 r^2)$$

Plug them into the charge density

$$\rho = -\frac{q}{4\pi}(-4\pi\delta^3(\vec{r}) + \frac{1}{2}\alpha^3 e^{-\alpha r}) = q\delta^3(\vec{r}) - \frac{q}{8\pi}\alpha^3 e^{-\alpha r}$$

The first term represents a point charge q at the origin and the second term is due to a continuous distributed volume charge. Note that the total charge is zero:

$$\int \rho d^3x = 0$$

Problem 1.6

(a) Applying Gauss's law to the plate with a Gaussian pillbox, one gets the electric field due to a flat surface charge distribution to be $\sigma/2\epsilon_0$, where $\sigma = Q/A$ is the surface charge density. The contributions from the two plates add up in between the two plates and cancel outside. The total electric field in between the two plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

The potential difference between the two plates

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E}\cdot d\vec{\ell} = \frac{Qd}{\epsilon_0 A}$$

The capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Assuming the inner sphere has charge Q and the outer has $-Q$, applying Gauss's law with a Gaussian sphere of radius r ($a < r < b$):

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

The potential difference

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E}\cdot d\vec{\ell} = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

The capacitance is therefore

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(c) Again assuming the inner cylinder has charge Q and the outer has $-Q$, applying Gauss's law with a cylindrical surface as the Gaussian surface:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0}\frac{\vec{r}}{r^2}$$

The potential difference

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0}\ln\frac{b}{a}$$

The capacitance per unit length

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln\frac{b}{a}}$$

Problem 1.7

From Gauss's law, the field due to one conductor is

$$\vec{E} = \frac{Q}{2\pi\epsilon_0}\frac{1}{r}$$

where Q is the charge per unit length and r is the perpendicular distance from the point of interest to the conductor. Along the perpendicular line joining the two conductors, the fields due to the two conductors are in the same direction. Therefore, the total field along the line is:

$$E = \frac{Q}{2\pi\epsilon_0}\frac{1}{r_+} + \frac{Q}{2\pi\epsilon_0}\frac{1}{r_-}$$

where r_+ and r_- are the perpendicular distances to the positively and negatively charged conductors respectively. $\vec{E}$ points from $+Q$ to $-Q$. The potential difference

$$V = \Phi_+ - \Phi_- = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0}\{\int_{a_1}^{d-a_2}\frac{1}{r_+}dr_+ + \int_{a_2}^{d-a_1}\frac{1}{r_-}dr_-\} = \frac{Q}{2\pi\epsilon_0}\ln\frac{(d-a_1)(d-a_2)}{a_1a_2}$$

The capacitance per unit length

$$C = \frac{Q}{V} = 2\pi\epsilon_0/\ln\frac{(d-a_1)(d-a_2)}{a_1a_2} \approx 2\pi\epsilon_0/\ln\frac{d^2}{(\sqrt{a_1a_2})^2} = \pi\epsilon_0/\ln\frac{d}{a}$$

where a is the geometrical mean of a_1 and a_2: $a = \sqrt{a_1a_2}$.

Problem 1.9

(a) *Parallel plate capacitor*

The negatively charged plate experiences a field of

$$E = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}$$

due to the positively charged plate, where Q is the total charge on the plate. Therefore, the attractive force between the two plates is

$$F = QE = \frac{Q^2}{2\epsilon_0 A}$$

Parallel cylinder capacitor
Again, one conductor experiences an electric field of

$$E \approx \frac{Q}{2\pi\epsilon_0}\frac{1}{d}$$

from the other conductor. Here Q is the charge per unit length. Therefore, the attractive force per unit length between them is

$$F = QE = \frac{Q^2}{2\pi\epsilon_0}\frac{1}{d}$$

(b) The force should be the same except that Q should be replaced by CV.
Parallel plate capacitor

$$F = \frac{\epsilon_0 A V^2}{2d^2}$$

Parallel cylinder capacitor

$$F = \frac{\pi\epsilon_0 V^2}{2d\ln(b/a)^2}$$