1 Problem 5.21

1.1

$$\begin{aligned}\vec{B}\cdot\vec{H}d^3x &= \left(\nabla\times\vec{A}\right)\cdot\vec{H}d^3x \\ &= \int\vec{A}\cdot\left(\nabla\times\vec{H}\right)d^3x \\ &= \int\vec{A}\cdot\underbrace{\vec{J}}_{0}d^3x = 0\end{aligned}$$

1.2

Starting from equation 5.72 in Jackson:

$$W = U = -\vec{m}\cdot\vec{B}$$

For a discrete number of point dipoles, we can write:

$$W = -\sum_{i<j}\vec{m}_i\cdot\vec{B}_j = -\frac{1}{2}\sum_{i\neq j}\vec{m}_i\cdot\vec{B}_j$$

Hence, for the continuous case, we can write:

$$\begin{aligned}W &= -\frac{1}{2}\int\vec{M}\cdot\vec{B}d^3x \\ &= -\frac{1}{2}\int\vec{M}\cdot\left[\mu_0\left(\vec{H}+\vec{M}\right)\right]d^3x \\ &= -\frac{\mu_0}{2}\int\vec{M}\cdot\vec{H}d^3x - \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x\end{aligned}$$

Note that the second integral is a constant which is independent of the position or orientation of the magnetized bodies (since we're integrating over all space).

Plugging in $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ into the above equation yields:

$$\begin{aligned}W &= -\frac{\mu_0}{2}\int\vec{M}\cdot\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right)d^3x - \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x \\ &= -\frac{1}{2}\cancelto{0}{\int\vec{M}\cdot\vec{B}d^3x} + \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x - \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x \\ &= \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x - \frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}d^3x\end{aligned}$$

where we've used the result from the previous part to show that the integral vanishes in the second step.

2 Problem 5.27

Let's pick our orientation such that the current through the inner wire is in the $\hat{z}$ direction:

$$\vec{J} = \hat{z} \begin{cases} \frac{I}{\pi b^2} & r < b \\ 0 & r > b \end{cases}$$

The current enclosed a loop of radius $r < b$ is:

$$I = 2\pi \int_0^r \frac{I}{\pi b^2} r dr = \frac{Ir^2}{b^2}$$

Using Ampère's Law, we can find the $\vec{B}$ field at all points in space:

$$\vec{B} = \hat{\varphi} \begin{cases} \frac{\mu I r}{2\pi b^2} & r < b \\ \frac{\mu_0 I}{2\pi r} & b < r < a \\ 0 & r > a \end{cases}$$

Finally, we use equation 5.157 in Jackson to find the self inductance:

$$\begin{aligned} L &= \frac{1}{I^2} \int \frac{\vec{B} \cdot \vec{B}}{\mu} d^3x \\ \frac{L}{\ell} &= \frac{1}{I^2} 2\pi \left[\frac{1}{\mu} \int_{r=0}^{b} \left(\frac{\mu I r}{2\pi b^2} \right)^2 r dr + \frac{1}{\mu_0} \int_{r=b}^{a} \left(\frac{\mu_0 I}{2\pi r} \right)^2 r dr \right] \\ \frac{L}{\ell} &= \frac{1}{I^2} 2\pi \left[\frac{\mu I^2}{4\pi^2 b^4} \frac{1}{4} b^4 + \frac{\mu_0 I^2}{4\pi^2} \ln\left(\frac{a}{b}\right) \right] \end{aligned}$$

$$\boxed{\frac{L}{\ell} = \frac{\mu}{8\pi} + \frac{\mu_0}{2\pi} \ln\left(\frac{a}{b}\right)}$$

If the inner conductor is a thin hollow tube, all the current will flow on the outside of the tube. Hence, there is no current (and hence no $\vec{B}$ field) for $r < b$. That is, $\vec{B}$ simplifies to:

$$\vec{B} = \hat{\varphi} \begin{cases} \frac{\mu_0 I}{2\pi r} & b < r < a \\ 0 & \text{otherwise} \end{cases}$$

And Equation 5.157 in Jackson becomes:

$$\begin{aligned} L &= \frac{1}{I^2} \int \frac{\vec{B} \cdot \vec{B}}{\mu} d^3x \\ \frac{L}{\ell} &= \frac{1}{I^2} 2\pi \left[\frac{1}{\mu_0} \int_{r=b}^{a} \left(\frac{\mu_0 I}{2\pi r} \right)^2 r dr \right] = \frac{1}{I^2} 2\pi \left[\frac{\mu_0 I^2}{4\pi^2} \ln\left(\frac{a}{b}\right) \right] \end{aligned}$$

$$\boxed{\frac{L}{\ell} = \frac{\mu_0}{2\pi} \ln\left(\frac{a}{b}\right)}$$