1 Problem 5.13

Starting with equation 5.32 in Jackson:

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int_V \vec{j}(\vec{r})\frac{1}{|\vec{r}-\vec{r}'|}d^3r' \\
&= \frac{\mu_0}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}\left(\sigma\underbrace{\vec{\omega}\times\vec{r}'}_{\omega r'\sin\theta'\hat{\varphi}'}\right)\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_\ell^{m*}(\theta',\varphi')Y_\ell^m(\theta,\varphi)\right)r'^2 d(\cos\theta')d\varphi'\Big|_{r'=a} \\
&= \frac{\mu_0}{4\pi}\sigma\omega a^3\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\int_{\varphi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}Y_\ell^{m*}(\theta',\varphi')Y_\ell^m(\theta,\varphi)\sin\theta'\hat{\varphi}'d(\cos\theta')d\varphi' \\
&= \mu_0\sigma\omega a^3\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell,m}(\theta,\varphi)\int_{\varphi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}Y_\ell^{m*}(\theta',\varphi')\sin\theta'\hat{\varphi}'d(\cos\theta')d\varphi'
\end{aligned} \tag{1}$$

Since $\hat{\varphi}' = -\sin\varphi'\hat{x} + \cos\varphi'\hat{y}$, the integral becomes:

$$\int_{\varphi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}Y_\ell^{m*}(\theta',\varphi')\left(-\sin\theta'\sin\varphi'\hat{x}+\sin\theta'\cos\varphi'\hat{y}\right)d(\cos\theta')d\varphi'$$

We know that $\sin\theta'\sin\varphi' = \sin\theta\left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right) = \frac{1}{i}\sqrt{\frac{2\pi}{3}}\left[-Y_1^1(\theta',\varphi') - Y_1^{-1}(\theta',\varphi')\right]$ and $\sin\theta'\cos\varphi' = \sin\theta\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right) = \sqrt{\frac{2\pi}{3}}\left[-Y_1^1(\theta',\varphi') + Y_1^{-1}(\theta',\varphi')\right]$. By exploiting orthogonality, the integral becomes:

$$\sqrt{\frac{2\pi}{3}}\int_{\varphi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}Y_\ell^{m*}(\theta',\varphi')\left[-i\left(Y_1^l(\theta',\varphi')+Y_1^{-1}(\theta',\varphi')\right)\hat{x}+\left(-Y_1^l(\theta',\varphi')+Y_1^{-1}(\theta',\varphi')\right)\hat{y}\right] d(\cos\theta')d\varphi'$$

$$=\sqrt{\frac{2\pi}{3}}\left[-i\left(\delta_1^1+\delta_1^{-1}\right)\hat{x}+\left(-\delta_1^1+\delta_1^{-1}\right)\hat{y}\right]$$

Plugging this into equation (1) yields:

$$\vec{A}(\vec{r}) = \mu_0\sigma\omega a^3\frac{1}{2(1)+1}\frac{r_<^1}{r_>^{(1)+1}}\sqrt{\frac{2\pi}{3}}\left[-i\left(Y_1^l(\theta,\varphi)+Y_1^{-1}(\theta,\varphi)\right)\hat{x}+\left(-Y_1^l(\theta,\varphi)+Y_1^{-1}(\theta,\varphi)\right)\hat{y}\right]$$

$$\vec{A}(\vec{r}) = \mu_0\sigma\omega a^3\frac{1}{2(1)+1}\frac{r_<^1}{r_>^{(1)+1}}\left[-\sin\theta\sin\varphi\hat{x}+\sin\theta\cos\varphi\hat{y}\right]$$

$$\vec{A}(\vec{r}) = \mu_0\sigma\omega a^3\frac{1}{3}\frac{r_<}{r_>^2}\sin\theta\underbrace{\left[-\sin\varphi\hat{x}+\cos\varphi\hat{y}\right]}_{\hat{\varphi}}$$

$$\vec{A}(\vec{r}) = \mu_0\sigma\omega a^3\frac{1}{3}\frac{r_<}{r_>^2}\sin\theta\hat{\varphi}$$

Inside the sphere:

$$\begin{aligned}\vec{A}_{\text{in}}(\vec{r}) &= \frac{1}{3}\mu_0\sigma\omega a^3 \frac{r}{a^2}\sin\theta\hat{\varphi} \\ &= \frac{1}{3}\mu_0\sigma\omega a r\sin\theta\hat{\varphi}\end{aligned}$$

Outside the sphere:

$$\begin{aligned}\vec{A}_{\text{out}}(\vec{r}) &= \frac{1}{3}\mu_0\sigma\omega a^3 \frac{a}{r^2}\sin\theta\hat{\varphi} \\ &= \frac{1}{3}\mu_0\sigma\omega \frac{a^4}{r^2}\sin\theta\hat{\varphi}\end{aligned}$$

$$\begin{aligned}\vec{B} &= \nabla\times\vec{A} \\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\hat{\theta}\end{aligned}$$

$$\begin{aligned}\vec{B}_{\text{in}} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{3}\mu_0\sigma\omega a r\sin^2\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{3}\mu_0\sigma\omega a r^2\sin\theta\right)\hat{\theta} \\ &= \frac{1}{r\sin\theta}\frac{1}{3}\mu_0\sigma\omega a r 2\sin\theta\cos\theta\hat{r} - \frac{1}{r}\frac{1}{3}\mu_0\sigma\omega a 2r\sin\theta\hat{\theta} \\ &= \frac{2}{3}\mu_0\sigma\omega a\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\end{aligned}$$

$$\begin{aligned}\vec{B}_{\text{out}} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^2}\sin^2\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r}\sin\theta\right)\hat{\theta} \\ &= \frac{1}{r\sin\theta}\frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^2}2\sin\theta\cos\theta\hat{r} - \frac{1}{r}\frac{1}{3}\mu_0\sigma\omega\left(-\frac{a^4}{r^2}\right)\sin\theta\hat{\theta} \\ &= \frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\end{aligned}$$

2 Problem 5.15

2.1

First, we use Ampère's Law to determine $\vec{B}$ for a single wire carrying current $I\hat{z}$:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{end}}$$
$$B 2\pi r = \mu_0 I$$
$$\implies \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

where the direction of $\vec{B}$ comes from applying the right-hand rule. Since $\vec{H} = \vec{B}/\mu_0$:

$$\vec{H} = \frac{I}{2\pi r}\hat{\varphi}$$

The magnetic scalar potential Φ_M is given by $\vec{H} = -\nabla\Phi_M$:

$$\vec{H} = -\nabla\Phi_M$$
$$\frac{I}{2\pi r}\hat{\varphi} = -\left(\cancel{\frac{\partial\Phi_M}{\partial r}}\hat{r} + \frac{1}{r}\frac{\partial\Phi_M}{\partial\varphi}\hat{\varphi} + \cancel{\frac{\partial\Phi_M}{\partial z}}\hat{z}\right)$$
$$\frac{I}{2\pi} = -\frac{\partial\Phi_M}{\partial\varphi}$$

$$\implies \Phi_M = -\frac{I}{2\pi}\varphi = -\frac{I}{2\pi}\arctan\left(\frac{y}{x}\right)$$

Now, the magnetic scalar potential of two parallel wires with currents $\pm I\hat{z}$ located at $x = \pm d/2$, $y = 0$ is:

$$\Phi_M = -\frac{I}{2\pi}\arctan\left(\frac{y}{x - d/2}\right) + \frac{I}{2\pi}\arctan\left(\frac{y}{x + d/2}\right)$$
$$= -\frac{I}{2\pi}\left[\arctan\left(\frac{y}{x - d/2}\right) - \arctan\left(\frac{y}{x + d/2}\right)\right]$$

Using the formula $\arctan u + \arctan v = \arctan\left(\frac{u+v}{1-uv}\right)$:

$$\begin{aligned}
\Phi_M &= -\frac{I}{2\pi}\arctan\left(\frac{\frac{y}{x-d/2}-\frac{y}{x+d/2}}{1+\frac{y}{x-d/2}\cdot\frac{y}{x+d/2}}\right) \\
&= -\frac{I}{2\pi}\arctan\left(\frac{y(x+d/2)-y(x-d/2)}{(x-d/2)(x+d/2)+y^2}\right) \\
&= -\frac{I}{2\pi}\arctan\left(\frac{yd}{x^2-d^2/4+y^2}\right) \\
&= -\frac{I}{2\pi}\arctan\left(\frac{r\sin\varphi d}{r^2\cos^2\varphi-d^2/4+r^2\sin^2\varphi}\right) \\
&= -\frac{I}{2\pi}\arctan\left(\frac{rd}{r^2-d^2/4}\sin\varphi\right) \\
&= -\frac{I}{2\pi}\arctan\left(\frac{d/r}{1-\cancelto{0}{(d/r)^2/4}}\sin\varphi\right) \\
&\approx -\frac{I}{2\pi}\arctan\left((d/r)\sin\varphi\right) \\
&\approx -\frac{I}{2\pi}\frac{d}{r}\sin\varphi
\end{aligned} \tag{2}$$

where we've used the fact that $\arctan x \approx x$.

2.2

We know that Φ_M must satisfy Laplace's equation, $\nabla^2\Phi_M = 0$. First, we want to find the general of Φ_M using separation of variables. We'll start out by assuming that $\Phi_M = \Phi_r(r)\Phi_\varphi(\varphi)$. Note that we're assumed that Φ_M has no z-dependence.

$$\begin{aligned}
\nabla^2\left(\Phi_r\Phi_\varphi\right) &= 0 \\
\Phi_\varphi\frac{1}{r}\frac{\partial}{\partial r}\left(r\Phi_r'\right)+\Phi_r\frac{1}{r^2}\Phi_\varphi'' &= 0 \\
\underbrace{\frac{1}{\Phi_r}r\left(\Phi_r'+r\Phi_r''\right)}_{k}+\underbrace{\frac{\Phi_\varphi''}{\Phi_\varphi}}_{-k} &= 0
\end{aligned}$$

$$\begin{aligned}
\Phi_\varphi'' = -k\Phi_\varphi &\implies \Phi_\varphi = A\sin(k\varphi)+B\cos(k\varphi) \\
r^2\Phi_r''+r\Phi_p' = k\Phi_r &\implies \Phi_r = Cr^\ell + Dr^{-\ell}
\end{aligned}$$

$$\therefore \Phi_M = \left(Cr^\ell + Dr^{-\ell}\right)\left(A\sin(k\varphi)+B\cos(k\varphi)\right)$$

Inside the cylinder, Φ_M must agree with equation (2) for $r \ll a$. By inspection, $B = 0$, $k = 1$, $\ell = 1$. We don't need to set $C = 0$ since $Cr^\ell \to 0$ when $r \ll a$. Hence:

$$\Phi_M = \left(C'r + D'r^{-1}\right)\sin(\varphi)$$

For $r < a$, D' is obviously $-Id/2\pi$. For $r > b$, $C' = 0$ because r blows up as $r \to \infty$. Hence:

$$\Phi_M = \begin{cases} \left(Ar - \frac{Id}{2\pi}r^{-1}\right)\sin\varphi & r < a \\ \left(Br + Cr^{-1}\right)\sin\varphi & a < r < b \\ Dr^{-1}\sin\varphi & r > b \end{cases}$$

Using the fact that $\vec{H} = -\nabla\Phi$, we can find $\vec{H}$:

$$H_r = -\frac{\partial \Phi_M}{\partial r} = \begin{cases} -\left(A + \frac{Id}{2\pi}r^{-2}\right)\sin\varphi & r < a \\ -\left(B - Cr^{-2}\right)\sin\varphi & a < r < b \\ Dr^{-2}\sin\varphi & r > b \end{cases}$$

$$H_\varphi = -\frac{1}{r}\frac{\partial \Phi_M}{\partial \varphi} = \begin{cases} -\left(A - \frac{Id}{2\pi}r^{-2}\right)\cos\varphi & r < a \\ -\left(B + Cr^{-2}\right)\cos\varphi & a < r < b \\ -Dr^{-2}\cos\varphi & r > b \end{cases}$$

Applying the boundary condition on H_r (the first of equations 5.89 in Jackson) at the boundaries ($r = a$, $r = b$):

$$-\mu_0\left(A + \frac{Id}{2\pi}a^{-2}\right)\sin\varphi = -\mu_r\mu_0\left(B - Ca^{-2}\right)\sin\varphi$$

$$A + \frac{Id}{2\pi}a^{-2} = \mu_r\left(B - Ca^{-2}\right) \tag{3}$$

$$-\mu_r\mu_0\left(B - Cb^{-2}\right)\sin\varphi = \mu_0 Db^{-2}\sin\varphi$$

$$\mu_r\left(-B + Cb^{-2}\right) = Db^{-2} \tag{4}$$

Applying the boundary condition on H_φ (the second of equations 5.89 in Jackson) at the boundaries ($r = a$, $r = b$):

$$-\left(A - \frac{Id}{2\pi}a^{-2}\right)\cos\varphi = -\left(B + Ca^{-2}\right)\cos\varphi$$

$$A - \frac{Id}{2\pi}a^{-2} = B + Ca^{-2} \tag{5}$$

$$-\left(B + Cb^{-2}\right)\cos\varphi = -Db^{-2}\cos\varphi$$

$$B + Cb^{-2} = Db^{-2} \tag{6}$$

We have four boundary condition equations – equations (3), (4), (5), (6). We will write these equations in matrix form to solve for A, B, C, and D:

$$\begin{bmatrix} 1 & -\mu_r & \mu_r a^{-2} & 0 \\ 0 & -\mu_r & \mu_r b^{-2} & -b^{-2} \\ 1 & -1 & -a^{-2} & 0 \\ 0 & 1 & b^{-2} & -b^{-2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{Id}{2\pi} a^{-2} \\ 0 \\ \frac{Id}{2\pi} a^{-2} \\ 0 \end{bmatrix}$$

Using the following Maple code, we solved this linear equation:

```
AA := matrix([[1, -mr, mr*a^(-2), 0], [0, -mr, mr*b^(-2), -b^(-2)],\
[1, -1, -a^(-2), 0], [0, 1, b^(-2), -b^(-2)]]);
bb := matrix([[-I*d*a^(-2)/(2*Pi)], [0], [I*d*a^(-2)/(2*Pi)], [0]]);
simplify(multiply(inverse(AA),bb));
```

The resulting values of A, B, C, and D are:

$$\begin{aligned} A &= -\frac{Id\left(-a^2 + b^2 + a^2\mu_r^2 - \mu_r^2 b^2\right)}{2a^2\pi\xi} \\ B &= -\frac{d(\mu_r - 1)}{\pi\xi} \\ C &= -\frac{db^2(\mu_r + 1)}{\pi\xi} \\ D &= -\frac{2I\mu_r b^2 d}{\pi\xi} \end{aligned}$$

where $\xi = b^2(\mu_r + 1)^2 - a^2(\mu_r - 1)^2$.

Outside the cylinder the magnetic scalar potential is:

$$\begin{aligned} \Phi_M &= Dr^{-1}\sin\varphi \\ &= \left(\frac{2I\mu_r b^2 d}{\pi\xi}\right) r^{-1}\sin\varphi \\ &= -\frac{Id\sin\varphi}{2\pi r}\underbrace{\frac{4\mu_r b^2}{\xi}}_{F} \end{aligned}$$

Φ_M in this region is (approximately) that of a two-dimensional dipole, scaled by the factor F.

This problem is related to problem 5.14, where the magnetic field is due to the wires instead of due to an external source.

2.3

For $\mu_r \gg 1$ and $b = a + t$, F becomes:

$$\begin{aligned} F &\approx \frac{4\mu_r b^2}{b^2\mu_r^2 - a^2\mu_r^2} \\ &= \frac{4b^2}{\mu_r\left(b^2 - (b-2t)^2\right)} \\ &\approx \frac{2b}{\mu_r t} \end{aligned}$$

For $\mu_r = 200$, $b = 1.25$ cm, $t = 0.3$ cm:

$$F \approx \frac{2(1.25 \text{ cm})}{(200)(0.3 \text{ cm})} = 0.042$$

3 Problem 5.19

We're given that $\vec{M} = M_0\hat{z}$. We start with Jackson 5.100 in Jackson:

$$\begin{aligned} \Phi_M(\vec{x}) &= -\frac{1}{4\pi}\int_V \frac{\cancelto{0}{\nabla' \cdot \vec{M}(\vec{x}')}}{|\vec{x}-\vec{x}'|}d^3x' + \frac{1}{4\pi}\oint_S \frac{\vec{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da' \\ &= \frac{1}{4\pi}\oint_{\text{top}} \frac{\hat{z}\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_{\text{bottom}} \frac{-\hat{z}\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da' \\ &= \frac{1}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a} \frac{M_0}{\sqrt{\rho'^2 + (z-L/2)^2}}\rho' d\rho' d\varphi' - \frac{1}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a} \frac{M_0}{\sqrt{\rho'^2 + (z+L/2)^2}}\rho' d\rho' d\varphi' \\ &= \frac{1}{2}\int_{\rho'=0}^{a} \frac{M_0}{\sqrt{\rho'^2 + (z-L/2)^2}}\rho' d\rho' - \frac{1}{2}\int_{\rho'=0}^{a} \frac{M_0}{\sqrt{\rho'^2 + (z+L/2)^2}}\rho' d\rho' \\ &= \frac{M_0}{2}\left[\sqrt{a^2 + (z-L/2)^2} - \left|z - \frac{L}{2}\right|\right] - \frac{M_0}{2}\left[\sqrt{a^2 + (z+L/2)^2} - \left|z + \frac{L}{2}\right|\right] \\ &= \frac{M_0}{2}\left[\sqrt{a^2 + (z-L/2)^2} - \sqrt{a^2 + (z+L/2)^2} - \left|z - \frac{L}{2}\right| + \left|z + \frac{L}{2}\right|\right] \end{aligned}$$

where the integral was evaluated using the Maple command
```
Q:=r/sqrt(r^2+(z-L/2)^2); int(Q,r=0..a);
```

On the axis of the cylinder, $\vec{H}$ lies only in the z-direction:

$$H_z = -\frac{\partial \Phi_M}{\partial z} = -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z-L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z+L/2)^2}} - \frac{\partial}{\partial z}\left|z - \frac{L}{2}\right| + \frac{\partial}{\partial z}\left|z + \frac{L}{2}\right|\right]$$

Note that the derivative of $|x|$ is $+1$ if $x > 0$, and is -1 if $x < 0$. This is the "sgn" function. Hence:

$$
\begin{aligned}
H_z = -\frac{\partial \Phi_M}{\partial z} &= -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} - \text{sgn}\left(z - \frac{L}{2}\right) + \text{sgn}\left(z + \frac{L}{2}\right)\right] \\
= -\frac{\partial \Phi_M}{\partial z} &= -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} + 2\text{Heaviside}\left(\frac{L}{2} - |z|\right)\right]
\end{aligned}
$$

Where "Heaviside" is the Heaviside step function.

Noting that $M_z = M_0 \text{Heaviside}\left(\frac{L}{2} - |z|\right)$:

$$
\begin{aligned}
\vec{B} &= \mu_0 \left(H_z + M_z\right) \\
&= -\mu_0 \frac{\partial \Phi_M}{\partial z} = -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} + 2\text{Heaviside}\left(\frac{L}{2} - |z|\right)\right] \\
&\qquad + \mu_0 M_0 \text{Heaviside}\left(\frac{L}{2} - |z|\right) \\
&= -\mu_0 \frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}}\right]
\end{aligned}
$$

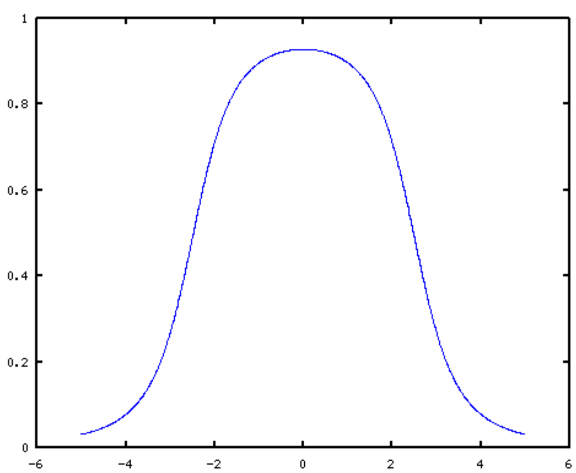

Figure 1: z vs. $B_z/\mu_0 M_0$

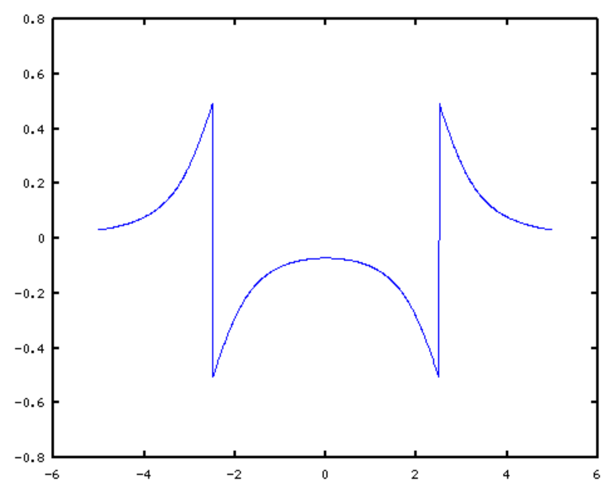

Figure 2: z vs. H_z/M_0