1 Problem 4.9

1.1

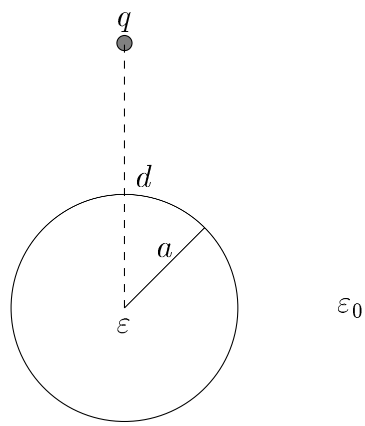

Figure 1: Setup for problem 4.9

Using the fact that we have azimuthal symmetry, we have inside the sphere:

$$\Phi_{\text{in}}(\vec{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{1}$$

And outside the sphere:

$$\Phi_{\text{out}}(\vec{r}) = \Phi_q + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

where Φ_q is the potential due to the charge q.

$$\Phi_q = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$

Because r' only points to the single point charge along the z-axis, $\gamma = \theta$. Therefore:

$$\Phi_{\text{out}}(\vec{r}) = \sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{r_<^l}{r_>^{l+1}} + B_l r^{-(l+1)}\right)P_l(\cos\theta) \tag{2}$$

Now, we need to apply the following boundary conditions:

$$-\frac{1}{a}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\bigg|_{r=a} = -\frac{1}{a}\frac{\partial\Phi_{\text{out}}}{\partial\theta}\bigg|_{r=a} \tag{3}$$

$$-\varepsilon\frac{\partial\Phi_{\text{in}}}{\partial r}\bigg|_{r=a} = -\varepsilon_0\frac{\partial\Phi_{\text{out}}}{\partial r}\bigg|_{r=a} \tag{4}$$

Applying eqation (3) yields:

$$-\frac{1}{a}\sum_{l=0}^{\infty} A_l r^l \dot{P}_l(\cos\theta)(-\sin\theta)\Bigg|_{r=a} = -\frac{1}{a}\sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{r_<^l}{r_>^{l+1}} + B_l r^{-(l+1)}\right)\dot{P}_l(\cos\theta)(-\sin\theta)\Bigg|_{r=a}$$

$$\sum_{l=0}^{\infty} A_l a^l = \sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{a^l}{d^{l+1}} + B_l a^{-(l+1)}\right)$$

$$A_l = \frac{q}{4\pi\varepsilon_0} d^{-(l+1)} + B_l a^{-(2l+1)} \tag{5}$$

Applying eqation (4) yields:

$$-\varepsilon\sum_{l=0}^{\infty} A_l l r^{l-1} P_l(\cos\theta)\Bigg|_{r=a} = -\varepsilon_0\sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{l r^{l-1}}{d^{l+1}} + B_l(-l-1)r^{-(l+2)}\right)P_l(\cos\theta)\Bigg|_{r=a}$$

$$\varepsilon\sum_{l=0}^{\infty} A_l l a^{l-1} = \varepsilon_0\sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{l a^{l-1}}{d^{l+1}} + B_l(-l-1)a^{-(l+2)}\right)$$

$$A_l = \frac{q}{4\pi\varepsilon} d^{-(l+1)} - \frac{\varepsilon_0}{\varepsilon} B_l \frac{l+1}{l} a^{-(2l+1)} \tag{6}$$

Equating equations (5) and (6):

$$\frac{q}{4\pi\varepsilon_0} d^{-(l+1)} + B_l a^{-(2l+1)} = \frac{q}{4\pi\varepsilon} d^{-(l+1)} - \frac{\varepsilon_0}{\varepsilon} B_l \frac{l+1}{l} a^{-(2l+1)}$$

$$B_l a^{-(2l+1)}\left(1 + \frac{\varepsilon_0}{\varepsilon}\frac{l+1}{l}\right) = \frac{q}{4\pi} d^{-(l+1)}\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0}\right)$$

$$B_l = \frac{q}{4\pi\varepsilon_0}\frac{a^{2l+1}}{d^{l+1}}\frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right) l}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)}$$

Plugging B_l into equation (5):

$$A_l = \frac{q}{4\pi\varepsilon_0} d^{-(l+1)} + \frac{q}{4\pi\varepsilon_0} d^{-(l+1)}\frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right) l}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)}$$

$$A_l = \frac{q}{4\pi\varepsilon_0} d^{-(l+1)}\left[1 + \frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right) l}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)}\right]$$

$$A_l = \frac{q}{4\pi\varepsilon} d^{-(l+1)}\frac{2l+1}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)}$$

Plugging these expressions for A_l and B_l into equations (1) and (2):

$$\boxed{\begin{aligned}\Phi_{\text{in}}(\vec{r}) &= \frac{q}{4\pi\varepsilon d}\sum_{l=0}^{\infty}\left(\frac{r}{d}\right)^l \frac{2l+1}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)} P_l(\cos\theta) \\ \Phi_{\text{out}}(\vec{r}) &= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} + \frac{a^{2l+1}}{(rd)^{l+1}}\frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right) l}{l + \frac{\varepsilon_0}{\varepsilon}(l+1)}\right) P_l(\cos\theta)\end{aligned}}$$

1.2

For $r/d \ll 1$, $l \geq 2$ terms are negligible. Thus, $\Phi_{\text{in}}(\vec{r})$ becomes:

$$\begin{aligned}\Phi_{\text{in}}(\vec{r}) &\approx \frac{q}{4\pi\varepsilon_0 d} + \frac{q}{4\pi\varepsilon d^2}\frac{3}{1+2\frac{\varepsilon_0}{\varepsilon}}\underbrace{r\cos\theta}_{z} \\ &= \frac{q}{4\pi\varepsilon_0 d} + \frac{q}{4\pi\varepsilon d^2}\frac{3}{1+2\frac{\varepsilon_0}{\varepsilon}}z\end{aligned}$$

Because $\vec{E} = -\nabla\Phi$:

$$\boxed{E = -\frac{q}{4\pi\varepsilon d^2}\frac{3}{1+2\frac{\varepsilon_0}{\varepsilon}}\hat{z}}$$

1.3

Our solution for Φ_{in} in part a is:

$$\begin{aligned}\Phi_{\text{in}}(\vec{r}) &= \frac{q}{4\pi\varepsilon d}\sum_{l=0}^{\infty}\left(\frac{r}{d}\right)^l \frac{2l+1}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}P_l(\cos\theta) \\ &= \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty}\left(\frac{r}{d}\right)^l \frac{2l+1}{\frac{\varepsilon}{\varepsilon_0}l+(l+1)}P_l(\cos\theta)\end{aligned}$$

For $\varepsilon/\varepsilon_0 \to \infty$, all the terms in the series go to zero, except for the $l = 0$ term:

$$\boxed{\Phi_{\text{in}} = \frac{q}{4\pi\varepsilon_0 d}}$$

For $\varepsilon/\varepsilon_0 \to \infty$, our solution for Φ_{out} in part a becomes:

$$\Phi_{\text{out}}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{a}{rd} + \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{(rd)^{l+1}}\right)P_l(\cos\theta)$$

Note that the extra term out front comes due to the fact that $\varepsilon_0/\varepsilon$ doesn't vanish when it multiplies l when $l = 0$. However, $\varepsilon_0/\varepsilon$ vanishes for all other terms in the series.

$$\Phi_{\text{out}}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{a}{rd} + \frac{q}{4\pi\varepsilon_0}\underbrace{\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)}_{1/|\vec{r}-d\hat{z}|} - \frac{qa}{4\pi\varepsilon_0}\underbrace{\sum_{l=0}^{\infty}\frac{a^{2l}}{(rd)^{l+1}}P_l(\cos\theta)}_{1/|a^2\hat{z}-d\vec{r}|}$$

$$\boxed{\Phi_{\text{out}}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{a}{rd} + \frac{1}{|\vec{r}-d\hat{z}|} - \frac{a/d}{\left|\frac{a^2}{d}\hat{z}-\vec{r}\right|}\right]}$$

$\Phi_{\text{out}}(\vec{r})$ agrees with equation 2.8 in Jackson. Note that we have $\Phi_{\text{in}}(a) = \Phi_{\text{out}}(a)$, as expected, and $\Phi_{\text{in}}(\vec{r})$ is constant, as expected, since the potential must remain constant inside a conductor. Hence, our solution for part a reduces to that of a conducting sphere in the limit $\varepsilon/\varepsilon_0 \to \infty$.

2 Problem 5.3

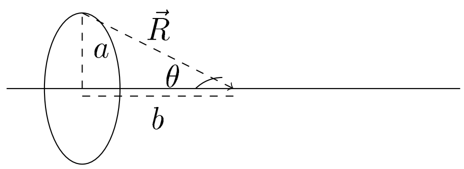

Figure 2: Single loop

Starting with the Biot-Savart Law for a loop with radius a and current I:

$$B = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{R}}{|\vec{R}|^3}$$

Noting that $d\vec{l} \times \vec{R} = dlR\sin\theta = dlR(a/R) = dla$:

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi} \int \frac{a}{R^3} dl \\ &= \frac{\mu_0 I}{4\pi} \frac{a}{R^3} (2\pi a) \\ &= \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + b^2)^{3/2}} \end{aligned}$$

For N loops squished together, B just becomes:

$$B = \frac{\mu_0 N I}{2} \frac{a^2}{(a^2 + b^2)^{3/2}}$$

To account for the rings to the left of the observation point, we integrate from 0 to c:

$$B_{\text{left}} = \int_0^c \frac{\mu_0 N I}{2} \frac{a^2}{(a^2 + b^2)^{3/2}} db$$

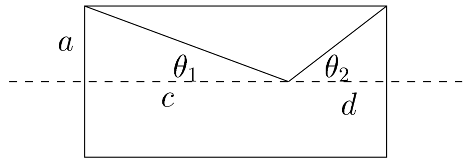

Figure 3:

Using the substitution $b = a\tan\theta$, $db = a\sec^2\theta d\theta$:

$$
\begin{aligned}
B_{\text{left}} &= \int_0^{\arctan(c/a)} \frac{\mu_0 NI}{2} \frac{a^2}{a^3\Bigg(\underbrace{1+\tan^2\theta}_{\sec^2\theta}\Bigg)^{3/2}} a\sec^2\theta d\theta \\
&= \frac{\mu_0 NI}{2}\int_0^{\arctan(c/a)} \cos\theta d\theta \\
&= \frac{\mu_0 NI}{2}\sin\left[\arctan\left(\frac{c}{a}\right)\right] \\
&= \frac{\mu_0 NI}{2}\frac{c}{\sqrt{a^2+c^2}} = \frac{\mu_0 NI}{2}\cos\theta_1
\end{aligned}
$$

To account for the rings to the right of the observation point, we integrate from 0 to d:

$$B_{\text{left}} = \int_0^d \frac{\mu_0 NI}{2}\frac{a^2}{(a^2+b^2)^{3/2}} db$$

Using the substitution $b = a\tan\theta$, $db = a\sec^2\theta d\theta$:

$$
\begin{aligned}
B_{\text{left}} &= \int_0^{\arctan(d/a)} \frac{\mu_0 NI}{2} \frac{a^2}{a^3\Bigg(\underbrace{1+\tan^2\theta}_{\sec^2\theta}\Bigg)^{3/2}} a\sec^2\theta d\theta \\
&= \frac{\mu_0 NI}{2}\int_0^{\arctan(d/a)} \cos\theta d\theta \\
&= \frac{\mu_0 NI}{2}\sin\left[\arctan\left(\frac{d}{a}\right)\right] \\
&= \frac{\mu_0 NI}{2}\frac{d}{\sqrt{a^2+d^2}} = \frac{\mu_0 NI}{2}\cos\theta_2
\end{aligned}
$$

Finally, $B = B_{\text{left}} + B_{\text{right}}$:

$$\boxed{B = \frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2)}$$

3 Problem 5.6

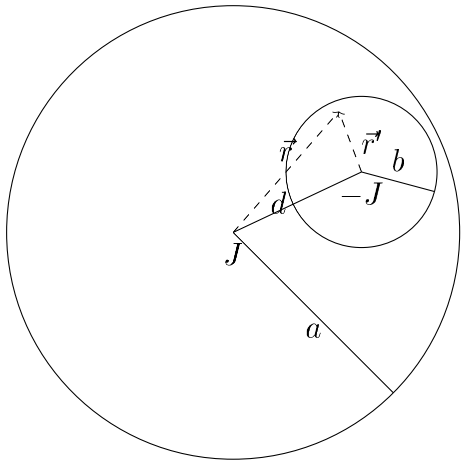

Figure 4: Setup for problem 5.6

We will consider two different systems and superimpose them (see figure 4).

1. A cylinder of radius a with current density $J\hat{z}$.
2. A cylinder of radius b with current density $-J\hat{z}$.

The B-field due to the cylinder in system 1 is:

$$\begin{aligned}
\oint \vec{B} \cdot d\vec{l} &= \mu_0 \int \vec{J} \cdot dS \\
B 2\pi r &= \mu_0 J \pi r^2 \\
\vec{B}_1 &= \frac{\mu_0}{2} J r \hat{\varphi} \\
&= \frac{\mu_0}{2} J r \left(\hat{z} \times \hat{r}\right)
\end{aligned}$$

The B-field due to the cylinder in system 2 is:

$$\begin{aligned}
\oint \vec{B} \cdot d\vec{l} &= \mu_0 \int \vec{J} \cdot dS \\
B 2\pi r' &= \mu_0 (-J) \pi r'^2 \\
\vec{B}_2 &= -\frac{\mu_0}{2} J r' \hat{\varphi}' \\
&= -\frac{\mu_0}{2} J r' \left(\hat{z} \times \hat{r}'\right)
\end{aligned}$$

Superimposing these two systems yields $\vec{B} = \vec{B}_1 + \vec{B}_2$:

$$\begin{aligned}\vec{B} &= \frac{\mu_0}{2} J \left(r\hat{z} \times \hat{r} - r'\hat{z} \times \hat{r}' \right) \\ &= \frac{\mu_0}{2} J\hat{z} \times \underbrace{\left(r\hat{r} - r'\hat{r}' \right)}_{\vec{d}}\end{aligned}$$

$$\boxed{\vec{B} = \frac{\mu_0}{2} J\hat{z} \times \vec{d}}$$