1 Problem 3.17

1.1

Equation 3.138 in Jackson:

$$\nabla^2 G(\vec{x},\vec{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

Substituting in the definition of the laplacian in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G(\vec{x},\vec{x}')}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 G(\vec{x},\vec{x}')}{\partial\varphi^2}+\frac{\partial^2 G(\vec{x},\vec{x}')}{\partial z^2} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z') \tag{1}$$

Now, we want to show that the Green function can be expressed in the following form:

$$G(\vec{x},\vec{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

First, we're going to confirm that this solution satisfies the above equation (that is, it's a valid Green function). Then, we're going to show that it is the Green function for the given setup. Plugging in the above Green function into equation (1):

$$\begin{aligned}\frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\right)\right. \\ \left. -\left(\frac{1}{\rho^2}m^2+\frac{n^2\pi^2}{L^2}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\right] = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')\end{aligned} \tag{2}$$

We will need the following identities of the Dirac delta function:

$$\delta(\varphi-\varphi') = \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \tag{3}$$

$$\delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \tag{4}$$

Hence, the right hand side of equation (2) becomes:

$$-\frac{8\pi}{\rho L}\delta(\rho-\rho')\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)$$

Thus, we have two summations equal to one another. Setting their coefficients equal to one another and canceling the exponential and sin terns from both sides of the equation:

$$\frac{4}{L}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\right)-\left(\frac{1}{\rho^2}m^2+\frac{n^2\pi^2}{L^2}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\right]$$
$$=-\frac{8\pi}{\rho L}\delta(\rho-\rho')$$

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\right)-\left(m^2+\frac{n^2\pi^2\rho^2}{L^2}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)=-2\delta(\rho-\rho')$$

Letting $x=-\frac{n\pi\rho}{L}$, this becomes the Bessel equation for $x\neq x'$. Clearly, $I_m(x_<)K_m(x_>)$ is a solution to this equation. Hence, we have shown that the proposed Green function is indeed a valid Green function.

To show that this is the Dirichlet Green function for the system at hand, we must show that is zero for $\vec{x}'$ is on the surfaces:

- The exponential function has no zeroes for finite φ'.
- $\sin\left(\frac{n\pi z'}{L}\right)$ is zero at $z=0,L$, as expected. Additionally, it is zero when z' is an integer multiple of L, which is allowed since using the method of images would effectively gives rise to an infinite number of surfaces at these places.
- I_0 has no zeroes. I_m $(m\neq 0)$ is zero only at $\rho'=0$.
- K_m has no zeroes for finite ρ'.

Finally, this Green function must not blow up anywhere. I_m and K_m are the only functions which blow up – they blow up at infinity and zero (respectively). However, since the argument of I_m is the lesser of ρ,ρ' and the argument of K_m is the greater of ρ,ρ' this Green function does not blow up at all.

Hence, we have shown that this is the Green function for the system at hand.

1.2

We want to show that the following Green function also solves the same setup:

$$G(\vec{x},\vec{x}')=2\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}dk e^{im(\varphi-\varphi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}$$

Again, we're going to show that this Green function satisfies equation (1). Plugging it in yields:

$$2\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk e^{im(\varphi-\varphi')}\frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}J_m(k\rho)J_m(k\rho')\right)\right.$$
$$\left.+\left(-\frac{1}{\rho^2}m^2+k^2\right)J_m(k\rho)J_m(k\rho')\right] = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

By substituting equation (3) into the right side of the above expression, we have two summations equal to one another. Equating the terms of each summation and cancelling the exponentials yields:

$$2\int_{-\infty}^{\infty} dk\frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}J_m(k\rho)J_m(k\rho')\right)\right.$$
$$\left.+\left(-\frac{1}{\rho^2}m^2+k^2\right)J_m(k\rho)J_m(k\rho')\right] = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(z-z')$$

Now, substituting the following identity into the right side of the above expression:

$$\delta(\rho-\rho') = \rho\int_0^{\infty} kJ_m(k\rho)J_m(k\rho')dk$$

we obtain two integrals equal to each other. Equating the integrands yields:

$$2\frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}J_m(k\rho)J_m(k\rho')\right)+\left(-\frac{1}{\rho^2}m^2+k^2\right)J_m(k\rho)J_m(k\rho')\right]$$
$$= -4\pi k\delta(z-z')$$

$$\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}J_m(k\rho)J_m(k\rho')\right)+\left(-m^2+\rho^2k^2\right)J_m(k\rho)J_m(k\rho')\right]$$
$$= -2\pi\rho^2k\delta(z-z')\frac{\sinh(kL)}{\sinh(kz_<)\sinh(k(L-z_>))}$$

Letting $x=\rho k$, this becomes the Bessel equation for $x\neq x'$, which is clearly solved by $J_m(k\rho)J_m(k\rho')$, indicating that this function satisfies equation 1 and is a valid Green function.

Again, to show that this Green function is zero on the surfaces, we note that $\sinh(kz')$ is zero at $z=0$ while $\sinh(k(L-z'))$ is zero at $z=L$. All of the functions are finite, so the Green function does not blow up anywhere.

2 Problem 3.26

We want to show that the following is the Green function for the volume between two concentric spheres with radii a and b (where $a < b$):

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$
$$+ \sum_{l=0}^{\infty} \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l} (rr')^l \frac{l}{l+1} \frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1} \left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right] P_l(\cos\gamma)$$

First, we note that the first summation is equal to $\frac{1}{|\vec{x}-\vec{x}'|}$ by equation 3.38 in Jackson. Hence, the second summation must satisfy Laplace's equation with respect to r' (equations 1.40, 1.41 in Jackson). We know that the general solution to Laplace's equation with azimuthal symmetry is:

$$F = \sum_{l=0}^{\infty} \left(A_l r'^l + B_l r'^{-(l+1)} \right) P_l(\cos\gamma)$$

So, we need to solve for A_l and B_l and show that F is equal to the second summation in the Green function which we're trying to validate.

Because this is a Neumann Green function, it must satisfy the Neumann boundary conditions:

$$\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S} = -\frac{1}{a^2 + b^2}$$

Note that $S = 4\pi(a^2 + b^2)$, which is the total surface area of all surfaces. Since the normal vector points away from the region of interest (and the region of interest is the area between the spheres), $n' = -r'$ for $r' = a$ and $n' = r'$ for $r' = b$. Also, noting that the normal derivative of the first summation of the Green function is zero for $r \neq r'$ yields the following boundary conditions:

$$\left.\frac{\partial F}{\partial r'}\right|_{r'=a} = \frac{1}{a^2 + b^2}$$
$$\left.\frac{\partial F}{\partial r'}\right|_{r'=b} = -\frac{1}{a^2 + b^2}$$

The left hand sides of the above equations are summations. We'll exploit the orthogonality of the Legendre polynomials to simplify the computation. Because $P_0(\cos\gamma) = 1$, these equations become:

$$\left.\frac{\partial F}{\partial r'}\right|_{r'=a} = \frac{1}{a^2 + b^2} P_0(\cos\gamma)$$
$$\left.\frac{\partial F}{\partial r'}\right|_{r'=b} = -\frac{1}{a^2 + b^2} P_0(\cos\gamma)$$

Because Legendre polynomials are orthogonal, the $l = 0$ term of the summation has to satisfy the above relations by itself, meaning that the remaining terms have to be zero. That is:

$$\frac{\partial}{\partial r'}\left(\frac{r'^l}{r^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}\right)\Bigg|_{r'=a} = 0$$
$$\frac{\partial}{\partial r'}\left(\frac{r^l}{r'^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}\right)\Bigg|_{r'=b} = 0$$

For $l > 0$. Evaluating the derivatives, these equations become:

$$l\frac{a^{l-1}}{r^{l+1}} + A_l l a^{l-1} + B_l(-l-1)a^{-l-2} = 0$$
$$-(l+1)\frac{r^{l-1}}{b^{l+2}} A_l l b^{l-1} + B_l(-l-1)b^{-l-2} = 0$$

$$\begin{bmatrix} la^{l-1} & (-l-1)a^{-l-2} \\ lb^{l-1} & (-l-1)b^{-l-2} \end{bmatrix}\begin{bmatrix} A_l \\ B_l \end{bmatrix} = \begin{bmatrix} -l\frac{a^{l-1}}{r^{l+1}} \\ (l+1)\frac{r^{l-1}}{b^{l+2}} \end{bmatrix}$$

Solving this linear equation yields:

$$A_l = \frac{1}{b^{2l+1} - a^{2l+1}}\left(\frac{a^{2l+1}}{r^{l+1}} + \frac{l+1}{l} r^{l-1}\right)$$
$$B_l = \frac{a^{2l+1}}{(b^{2l+1} - a^{2l+1})}\left(\frac{l}{l+1}\frac{b^{2l+1}}{r^{l+1}} + r^{l-1}\right)$$

Plugging these terms into:

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + A_l r'^l + B_l r'^{-(l+1)} \tag{5}$$

yields the radial Green function which we're validating (for $l > 0$).

2.1

As we noted above, the $l = 0$ term of the summation satisfies the Neumann boundary conditions by itself. That is:

$$\frac{\partial}{\partial r'}\left(\frac{r'^l}{r^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}\right)\Bigg|_{l=0, r'=a} = \frac{1}{a^2 + b^2}$$
$$\frac{\partial}{\partial r'}\left(\frac{r^l}{r'^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}\right)\Bigg|_{l=0, r'=b} = -\frac{1}{a^2 + b^2}$$

For $l > 0$. Evaluating the derivatives, these equations become:

$$-B_0 a^{-2} = \frac{1}{a^2+b^2}$$
$$-\frac{1}{b^2} - B_0 b^{-2} = -\frac{1}{a^2+b^2}$$

The A_0 term vanishes because it is multiplied by l (and $l = 0$). Solving for B_0 is trivial:

$$B_0 = -\frac{a^2}{a^2+b^2}$$

Plugging this into equation (5) for $l = 0$ yields:

$$g_0(r, r') = \frac{1}{r_>} + A_0(r) - \left(\frac{a^2}{a^2+b^2}\right) r'^{-1}$$

In order for $A_0(r)$, it must play no role in finding $\Phi(\vec{x})$. That is, the following must evaluate to zero:

$$\frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{r}')A_0(r)d^3r' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} A_0(r)da'$$

Because $A_0(r)$ does not depend on r', we can pull it out of both integrals:

$$A_0(r)\frac{1}{4\pi\varepsilon_0} \underbrace{\int_V \rho(\vec{r}')d^3r'}_{Q} + \frac{1}{4\pi} A_0(r) \oint_S \underbrace{\frac{\partial \Phi}{\partial n'}}_{-E_{n'}} da' = A_0(r)\frac{1}{4\pi}\left(\frac{Q}{\varepsilon_0} - \oint_S E_{n'} da'\right)$$

The term in parentheses is zero by Gauss' Law. Hence, $A_0(r)$ may be any arbitrary function of r, as long as it does not depend on r'.

3 Problem 3.27

Because there are no charges in this system, the first integral in equation 1.46 in Jackson is zero. Hence:

$$\Phi(\vec{x}) = \frac{1}{4\pi} \oint_S \underbrace{\frac{\partial \Phi}{\partial n'}}_{-E_r} G_N da'$$
$$= \frac{1}{4\pi} \oint_S (E_0 \cos\theta) \left(\sum_{l=0}^{\infty} g_l(r, r') P_l(\cos\gamma)\right) da'$$

Using the following identity:

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-\infty}^{\infty} Y^*_{l,m}(\theta', \varphi')Y_{l,m}(\theta, \varphi)$$

and the fact that $Y_{1,0}(\theta,\varphi)=\cos\theta$, $\Phi(\vec{x})$ becomes:

$$\Phi(\vec{x}) = \frac{E_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{4\pi}{2l+1}\oint_S Y_{1,0}(\theta,\varphi)g_l(r,r')Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)r'^2\sin\theta dr d\theta d\varphi$$

Because E_r is zero on the inner sphere, we only need to integrate over the outer sphere, Hence, $r'=b$. So, we're able to pull b^2, $g_l(r,b)$, and $Y_{l,m}(\theta,\varphi)$ out of the integral:

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{E_0}{\cancel{4\pi}}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\cancel{4\pi}}{2l+1}g_l(r,b)Y_{l,m}(\theta,\varphi)b^2\underbrace{\oint_S Y_{1,0}(\theta,\varphi)Y^*_{l,m}(\theta',\varphi')da'}_{\delta_{l=1,m=0}} \\
&= \frac{E_0}{3}g_1(r,b)\underbrace{Y_{1,0}(\theta,\varphi)}_{\cos\theta}b^2 \\
&= \frac{E_0}{3}\left(\frac{r}{b^2}+\frac{1}{b^3-a^3}\left[2(rb)+\frac{1}{2}\frac{(ab)^3}{(rb)^2}+a^3\left(\frac{r}{b^2}+\frac{b}{r^2}\right)\right]\right)b^2\cos\theta \\
&= \frac{E_0}{3}\left(r+\frac{1}{1-(a/b)^3}\frac{b^2}{b^3}\left[2rb+\frac{3}{2}\frac{a^3b}{r^2}+\frac{a^3r}{b^2}\right]\cos\theta\right) \\
&= \frac{E_0}{3}\frac{1}{1-(a/b)^3}\left[r\left(1-\left(\frac{a}{b}\right)^3\right)+2r+\frac{3}{2}\frac{a^3}{r^2}+\frac{a^3r}{b^3}\right]\cos\theta \\
&= \frac{E_0}{3}\frac{1}{1-(a/b)^3}\left[3r+\frac{3}{2}\frac{a^3}{r^2}\right]\cos\theta \\
&= E_0\frac{\cos\theta}{1-(a/b)^3}\left[r+\frac{a^3}{2r^2}\right]
\end{aligned}$$

$$\vec{E} = -\nabla\Phi(\vec{x})$$

We obtain the components of E using the definition of $\nabla\varphi$ in spherical coordinates:

$$\begin{aligned}
E_r &= -\frac{\partial\Phi}{\partial r} \\
&= -E_0\frac{\cos\theta}{1-(a/b)^3}\left[1-\frac{a^3}{r^3}\right] \\
E_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} \\
&= E_0\frac{\sin\theta}{1-(a/b)^3}\left[r+\frac{a^3}{2r^2}\right]
\end{aligned}$$

3.1

Converting $\Phi(\vec{x})$ to cylindrical coordinates:

$$\begin{aligned}
\Phi(\vec{x}) &= E_0 \frac{\cos\left(\cos^{-1}(z/\sqrt{\rho^2+z^2})\right)}{1-(a/b)^3} \left[\sqrt{\rho^2+z^2} + \frac{a^3}{2\left(\rho^2+z^2\right)}\right] \\
&= E_0 \frac{1}{1-(a/b)^3} \left[z + \frac{za^3}{2\left(\rho^2+z^2\right)^{3/2}}\right]
\end{aligned}$$

We obtain the components of E using the definition of $\nabla\varphi$ in cylindrical coordinates:

$$\begin{aligned}
E_\rho &= -\frac{\partial\Phi}{\partial\rho} \\
&= \frac{E_0}{1-(a/b)^3} \frac{3z\rho a^3}{2\left(\rho^2+z^2\right)^{5/2}} \\
E_z &= -\frac{\partial\Phi}{\partial z} \\
&= -E_0 \frac{1}{1-(a/b)^3} \left[1 + \frac{2\left(\rho^2+z^2\right)^{3/2} a^3 - 3z^2 a^3 \sqrt{\rho^2+z^2}}{2\left(\rho^2+z^2\right)^3}\right]
\end{aligned}$$