Problem 3.1

Two concentric spheres have radii a, b $(b > a)$ and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.
Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$ and $a \to 0$.

Solution to Laplace's equation with azimuthal symmetry:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)$$

Consider: $r = a$

$$\begin{aligned}\left(A_l a^l + B_l a^{-l-1}\right) \frac{2}{2l+1} &= V \underbrace{\int_0^1 P_l(\cos\theta) d(\cos\theta)}_{N_l} \\ &= V N_l\end{aligned}$$

Consider: $r = b$

$$\begin{aligned}\left(A_l b^l + B_l b^{-l-1}\right) \frac{2}{2l+1} &= V \int_{-1}^{0} P_1(\cos\theta) d(\cos\theta) \\ &= V \int_{1}^{0} P_l(-\cos\theta) d(-\cos\theta) \\ &= V \int_{0}^{1} P_l(-\cos\theta) d(\cos\theta) \\ &= V \int_{0}^{1} (-l)^l P_l(-\cos\theta) d(\cos\theta) \\ &= V (-1)^l N_l\end{aligned}$$

$$\begin{bmatrix} a^l & a^{-(l+1)} \\ b^l & b^{-(l+1)} \end{bmatrix} \begin{bmatrix} A_l \\ B_l \end{bmatrix} = \frac{2l+1}{2} V N_l \begin{bmatrix} 1 \\ (-1)^l \end{bmatrix}$$

Solve for $\begin{bmatrix} A_l \\ B_l \end{bmatrix}$:

$$\begin{aligned}
\begin{bmatrix} A_l \\ B_l \end{bmatrix} &= \frac{2l+1}{2} V N_l \begin{bmatrix} a^l & a^{-(l+1)} \\ b^l & b^{-(l+1)} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ (-1)^l \end{bmatrix} \\
&= \frac{2l+1}{2} \frac{V N_l}{a^l b^{-(l+1)} - b^l a^{-(l+1)}} \begin{bmatrix} b^{-(l+1)} & -a^{-(l+1)} \\ -b^l & a^l \end{bmatrix} \begin{bmatrix} 1 \\ (-1)^l \end{bmatrix} \\
&= \frac{2l+1}{2} \frac{V N_l}{a^l b^{-(l+1)} - b^l a^{-(l+1)}} \begin{bmatrix} b^{-(l+1)} + (-1)^{l+1} a^{-(l+1)} \\ -b^l + (-1)^l a^l \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Phi(r,\theta) = \sum_{l=0}^{\infty} & \left[\left(b^{-(l+1)} + (-1)^{l+1} a^{-(l+1)} \right) r^l + \left(-b^l + (-1)^l a^l \right) r^{-(l+1)} \right] \\
& \times \frac{2l+1}{2} \frac{V}{a^l b^{-(l+1)} - b^l a^{-(l+1)}} N_l P_l(\cos\theta)
\end{aligned}$$

We saw in class that that $N_l = \int_0^1 P_l(x) dx = 0$ for even even l (except for $l = 0$, where it is 1). For odd l, $N_{2l-1} = \frac{(-1)^{l-1}}{(2l-1)2^{2l}} \binom{2l}{l-1}$:

$$\begin{aligned}
\Phi(r,\theta) = \frac{V}{2} + \sum_{l=1}^{\infty} & \left[\left(b^{-2l} + a^{-2l} \right) r^{2l-1} + \left(-b^{2l-1} - a^{2l-1} \right) r^{-2l} \right] \\
& \times - \frac{4l+3}{2} \frac{V}{a^{2l-1} b^{-2l} - b^l a^{-2l}} \frac{1}{(2l-1)2^{2l}} \binom{2l}{l-1} P_{2l-1}(\cos\theta)
\end{aligned}$$

The terms up to $l = 4$ (i.e., $l = 0, 1, 3$) are:

$$\begin{aligned}
\Phi(r,\theta) = & \frac{V}{2} + \left[\left(b^{-2} + a^{-2} \right) r + \left(-b^2 - a \right) r^{-2} \right] \frac{7}{2} \frac{V}{ab^{-2} - ba^{-2}} \frac{1}{4} \binom{2}{0} P_1(\cos\theta) \\
& - \left[\left(b^{-4} + a^{-4} \right) r^3 + \left(-b^3 - a^3 \right) r^{-4} \right] \frac{11}{2} \frac{V}{a^3 b^{-4} - b^2 a^{-4}} \frac{1}{48} \binom{4}{1} P_3(\cos\theta)
\end{aligned}$$

$$\begin{aligned}
\Phi(r,\theta) = & \frac{V}{2} + \left[\left(b^{-2} + a^{-2} \right) r + \left(-b^2 - a \right) r^{-2} \right] \frac{7}{8} \frac{V}{ab^{-2} - ba^{-2}} \cos(\theta) \\
& - \left[\left(b^{-4} + a^{-4} \right) r^3 + \left(-b^3 - a^3 \right) r^{-4} \right] \frac{11}{48} \frac{V}{a^3 b^{-4} - b^2 a^{-4}} \left(5\cos^3\theta - 3\cos\theta \right)
\end{aligned}$$

For the limit as $b \to \infty$:

$$\lim_{b\to\infty} \Phi(r,\theta) = \frac{V}{2} - \sum_{l=1}^{\infty} r^{-2l} \frac{4l+3}{2} \frac{V}{(2l-1)2^{2l}} \binom{2l}{l-1} P_{2l-1}(\cos\theta)$$

For the limit as $a \to 0$:

$$\lim_{a\to\infty} \Phi(r,\theta) = \frac{V}{2} - \sum_{l=1}^{\infty} r^{2l-1} \frac{4l+3}{2} \frac{V}{(2l-1)2^{2l}} \binom{2l}{l-1} P_{2l-1}(\cos\theta)$$

1 Problem 3.3

A thin, flat, conducting, circular disc of radius R is located in the x-y plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance from the center of the disc:

1.1 Show that for $r > R$ the potential is:

$$\Phi(r,\theta,\varphi) = \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta)$$

We know we can start with:

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|} \tag{1}$$

We know that:

$$\frac{1}{\vec{r} - \vec{r}'} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

Because we're outside, $r > r'$. Using this fact and plugging the above equation into equation (1):

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\gamma) \tag{2}$$

We also know that:

$$\rho(r) = \frac{\sigma(r)}{r} \delta(\cos\theta)$$
$$\sigma(r) = \frac{\lambda}{\sqrt{R^2 - r^2}}$$

We just need to know λ. Since $\Phi = V$ on the disc, $\Phi(r = 0) = V$. Hence:

$$\begin{aligned}
\Phi(r=0) = V &= \int \frac{\rho(\vec{r}')}{4\pi\varepsilon_0 r'} d^3r' \\
&= \frac{\lambda\pi}{4\varepsilon_0} \\
\implies \lambda &= \frac{4V\varepsilon_0}{\pi}
\end{aligned}$$

Plugging λ into σ and plugging that into ρ:

$$\rho(r) = \frac{4V\varepsilon_0}{\pi r\sqrt{R^2 - r^2}}\delta(\cos\theta)$$

Plugging this into equation (2):

$$\begin{aligned}
\Phi(\vec{r}) &= \int \left(\frac{4V\varepsilon_0}{\pi r'\sqrt{R^2 - r'^2}}\delta(\cos\theta)\right)\frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma) \\
&= \frac{V}{\pi^2}\int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\theta)\delta(\cos\theta)\int_0^R r'^2 dr' \sum_{l=0}^{\infty}\frac{r'^{l-1}}{r^{l+1}\sqrt{R^2 - r'^2}}P_l(\cos\gamma) \\
&= \frac{V}{\pi^2}2\pi\sum_{l=0}^{\infty} r^{-(l+1)}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}}dr' \cos(\gamma)
\end{aligned}$$

For simplicity, let's pick $\gamma = \frac{\pi}{2}$. This will force $\vec{r}$ to always lie along the z-axis, which will give us an expression for $\Phi(\vec{z})$. Using the fact that $P_l(\cos\gamma) = P_l(0) = \frac{(-1)^n(2n-1)!!}{2^n n!}$, where $l = 2n$:

$$\Phi(\vec{z}) = \frac{2V}{\pi}\sum_{l=0}^{\infty} r^{-(l+1)}\frac{(-1)^n(2n-1)!!}{2^n n!}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}}dr'$$

We know that the general solution for azimuthal symmetry is:

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) \tag{3}$$

The general solution must agree with equation (1.1) along the z axis. To do this, we set $\theta = 0$ (and $P_l(\cos(0)) = P_l(1) = 1$):

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) \tag{4}$$

Equating equations (1.1) and (4) we find that:

$$\begin{aligned}
A_l &= 0 \\
B_l &= \frac{2V}{\pi}\frac{(-1)^n(2n-1)!!}{2^n n!}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}}dr'
\end{aligned}$$

Plugging this into equation (3):

$$\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\left(\frac{(-1)^n(2n-1)!!}{2^n n!}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}}dr'\right) r^{-(l+1)}P_l(\cos\theta)$$

Using an integral table, we find that $\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}dr'} = R^{2n+1}\frac{n!2^n}{(2n+1)!!}$ where $l = 2n$:

$$\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{l=0}^{\infty} r^{-(l+1)} R^{2n+1} \frac{(-1)^n(2n-1)!!}{(2n+1)!!} P_l(\cos\theta)$$
$$= \frac{2V}{\pi}\sum_{l=0}^{\infty} r^{-(l+1)} R^{2n+1} \frac{(-1)^n}{(2n+1)} P_l(\cos\theta)$$

Finally, converting all the indices to n using the relation $l = 2n$:

$$\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{n=0}^{\infty} r^{-(2n+1)} R^{2n+1} \frac{(-1)^n}{(2n+1)} P_{2n}(\cos\theta) \tag{5}$$
$$\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{2n+1} \frac{(-1)^n}{(2n+1)} P_{2n}(\cos\theta)$$
$$\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{2n} \frac{R}{r}\frac{(-1)^n}{(2n+1)} P_{2n}(\cos\theta)$$

$$\boxed{\Phi(\vec{r}) = \frac{2V}{\pi}\frac{R}{r}\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{2n} \frac{(-1)^n}{(2n+1)} P_{2n}(\cos\theta)}$$

1.2 Find the potential for $r < R$

At the boundary, $\Phi(\vec{r})$ for $r > R$ and $r < R$ must be equal. Hence:

$$A_l R^l = B_l R^{-(l+1)}$$
$$A_{2n} R^{2n} = B_{2n} R^{-(2n+1)}$$

Plugging in B_{2n} (the $r^{-(2n+1)}$ coefficient from equation (5)):

$$A_{2n} R^{2n} = \left(\frac{2V}{\pi}\cancel{R^{2n+1}}\frac{(-1)^n}{(2n+1)}\right)\cancel{R^{-(2n+1)}}$$
$$A_{2n} = \frac{2V}{\pi}\frac{(-1)^n}{(2n+1)} R^{-2n}$$

Plugging this into equation (3) (the general solution for azimuthal symmetry):

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left(\frac{2V}{\pi}\frac{(-1)^n}{(2n+1)} R^{-2n}\right) r^{2n} P_{2n}\cos(\theta)$$

$$\boxed{\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^n}{(2n+1)}\left(\frac{r}{R}\right)^{2n} P_{2n}\cos(\theta)}$$

1.3 Find the capacitance of the disk

In part 1, we found that:

$$V = \frac{\lambda\pi}{4\varepsilon_0}$$
$$\sigma(r) = \frac{\lambda}{\sqrt{R^2 - r^2}}$$

$$\begin{aligned} Q &= \int_{\varphi=0}^{2\pi} \int_{r=0}^{R} \frac{\lambda}{\sqrt{R^2 - r^2}} r\,dr \\ &= 2\pi\lambda \left[-\sqrt{R^2 - r^2} \right]_0^R \\ &= 2\pi\lambda R \end{aligned}$$

The capacitance is:

$$\begin{aligned} C &= \frac{Q}{V} \\ &= \frac{2\pi\lambda R}{\frac{\lambda\pi}{4\varepsilon_0}} \\ &= \boxed{8R\varepsilon_0} \end{aligned}$$