Problem 2.13

Two halves of a long hollow conducting cylinder of inner radius b are separated by small lengthwise gaps on each side, and are kept at dierent potentials V_1 and V_2

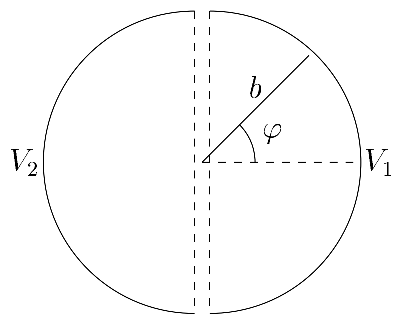

Figure 1: System for problem 2.13

2.13.a

Show that the potential inside is given by:

$$\Phi(\rho, \varphi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2 - \rho^2} \cos\varphi\right)$$

where φ is measured from a plane perpendicular to the plane through the gap.

Using the result from problem 2.12:

$$\begin{aligned}
\Phi(\rho,\varphi) =& \frac{1}{2\pi}\int_0^{2\pi} \Phi(\rho = b, \varphi) \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi)} d\varphi' \\
=& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} V_1 \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi)} d\varphi' + \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} V_2 \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi)} d\varphi' \\
=& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} V_1 \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi)} d\varphi' + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} V_2 \frac{b^2 - \rho^2}{b^2 + \rho^2 + 2b\rho\cos(\varphi' - \varphi)} d\varphi' \\
=& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \left[V_1 \frac{(b^2 - \rho^2)(b^2 + \rho^2 + 2b\rho\cos(\varphi' - \varphi))}{(b^2 + \rho^2)^2 - (2b\rho\cos(\varphi' - \varphi))^2} + V_2 \frac{(b^2 - \rho^2)(b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi))}{(b^2 + \rho^2)^2 - (2b\rho\cos(\varphi' - \varphi))^2} \right] d\varphi' \\
=& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(V_1 + V_2)(b^2 - \rho^2)(b^2 + \rho^2)}{(b^2 + \rho^2)^2 - 4b^2\rho^2\cos^2(\varphi' - \varphi)} + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(V_1 - V_2)\, 2b\rho\cos(\varphi' - \varphi)}{(b^2 + \rho^2)^2 - 4b^2\rho^2\cos^2(\varphi' - \varphi)} d\varphi' \\
=& \frac{1}{2\pi}\pi(V_1 + V_2) + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(V_1 - V_2)(b^2 - \rho^2)\, 2b\rho\cos(\varphi' - \varphi)}{b^4 + \rho^4 + 2b^4\rho^2 + (2b^2\rho^2 - 2b^2\rho^2) - 4b^2\rho^2\cos^2(\varphi' - \varphi)} d\varphi'
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2\pi}\pi\left(V_1+V_2\right)+\frac{1}{2\pi}\frac{2\left(V_1-V_2\right)\left(b^2-\rho^2\right)}{b^2-\rho^2}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\varphi\right)\\
=&\boxed{\frac{\left(V_1+V_2\right)}{2}+\frac{\left(V_1-V_2\right)}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\varphi\right)}
\end{aligned}
$$

2.13.b. Calculate the surface-charge density on each half of the cylinder

$$
\begin{aligned}
\sigma=-\varepsilon_0\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b}&=-\varepsilon_0\left.\frac{V_1-V_2}{\pi}\frac{1}{1+\left(\frac{2\rho b}{b^2-\rho^2}\cos\varphi\right)^2}\frac{\left(b^2-\rho^2\right)2b-2\rho b\left(-2\rho\right)}{\left(b^2-\rho^2\right)^2}\cos\varphi\right|_{\rho=b}\\
&=-\varepsilon_0\left.\frac{V_1-V_2}{\pi}\frac{4b\rho^2}{\left(b^2-\rho^2\right)^2+\left(2\rho b\cos\varphi\right)^2}\cos\varphi\right|_{\rho=b}\\
&=\boxed{-\varepsilon_0\frac{V_1-V_2}{\pi b\cos\varphi}}
\end{aligned}
$$

Problem 2.14

A variant of the preceding two-dimensional problem is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.

2.14.a.

Solve by means of series solution (2.71) and show that the potential inside the cylinder is:

$$
\Phi(\rho,\varphi)=\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin\left[(4n+2)\varphi\right]}{2n+1}
$$

$$
\Phi(\rho,\varphi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\left[a_n\rho^n\sin\left(n\varphi+\alpha_n\right)+b_n\rho^{-n}\sin\left(n\varphi+\beta_n\right)\right]
$$

Since we're finding the solution inside the cylinder $b_n=0$:

$$
\begin{aligned}
\Phi(\rho,\varphi)&=a_0+\sum_{n=1}^{\infty}a_n\rho^n\sin\left(n\varphi+\alpha_n\right)\\
&=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\rho^n\cos\left(n\varphi\right)+B_n\rho^n\sin\left(n\varphi\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
B_n b^n &= \frac{1}{2\pi}\int_0^{2\pi} \Phi(\rho = b, \varphi)\sin(n\varphi) \\
&= \frac{2}{2\pi}\left[\int_0^{\pi/2}(V)\sin(n\varphi) + \int_{\pi/2}^{\pi}(-V)\sin(n\varphi) + \int_{\pi}^{3\pi/2}(V)\sin(n\varphi) + \int_{3\pi/2}^{2\pi}(-V)\sin(n\varphi)\right] \\
&= \frac{1}{\pi}(-V)\left[\frac{1}{n}\cos(n\varphi)\Big|_0^{\pi/2} - \frac{1}{n}\cos(n\varphi)\Big|_{\pi/2}^{\pi} + \frac{1}{n}\cos(n\varphi)\Big|_{\pi}^{3\pi/2} - \frac{1}{n}\cos(n\varphi)\Big|_{3\pi/2}^{2\pi}\right] \\
&= \frac{1}{\pi}\left(-\frac{V}{n}\right)\left[\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right] + \left[\cos\left(\frac{3n\pi}{2}\right) - (-1)^n\right] - \left[1 - \cos\left(\frac{3n\pi}{2}\right)\right]\right] \\
&= \frac{1}{\pi}\left(\frac{V}{n}\right)\left[2 + 2\,(-1)^n + 4\cos\left(\frac{n\pi}{2}\right)\right] \\
&= \begin{cases} \frac{8V}{n\pi} & n = 2, 6, 10, 14, 18, 22\cdots = 4m-2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

All the A_n terms will integrate to zero because $\Phi(\beta = bm\varphi)$ is an odd function.

$$
\begin{aligned}
\Phi(\rho, \varphi) &= \sum_{m=1}^{\infty}\left(\frac{8V}{(4m-2)\pi}\right)\frac{1}{b^{4m-2}}\rho^{4m-2}\sin\left[(4m-2)\varphi\right] \\
&= \sum_{m=1}^{\infty}\left(\frac{\rho}{b}\right)^{4m+2}\left(\frac{4V}{(2m-1)\pi}\right)\sin\left[(4m-2)\varphi\right] \\
&= \boxed{\frac{4V}{\pi}\sum_{m=0}^{\infty}\left(\frac{\rho}{b}\right)^{4m+2}\frac{\sin\left[(4m+2)\varphi\right]}{2m+1}}
\end{aligned}
$$

2.14.b.

Sum the series and show that:

$$
\Phi(\rho, \varphi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin 2\varphi}{b^4 - \rho^4}\right)
$$

$$\begin{aligned}
\Phi(\rho,\varphi) &= \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin\left[(4n+2)\varphi\right]}{2n+1} \\
&= \frac{4V}{\pi}\text{Imag}\left[\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{e^{i(4n+2)\varphi}}{2n+1}\right] \\
&= \frac{4V}{\pi}\text{Imag}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)^{2n+1}\right] \\
&= \frac{4V}{\pi}\text{Imag}\left[\ln\underbrace{\left(\frac{1+\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}{1-\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}\right)}_{R}\right]
\end{aligned}$$

$$\begin{aligned}
\text{Real}[R] &= \frac{1}{2}\left[\left(\frac{1+\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}{1-\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}\right)+\left(\frac{1+\left(\frac{\rho^2}{b^2}e^{-i2\varphi}\right)}{1-\left(\frac{\rho^2}{b^2}e^{-i2\varphi}\right)}\right)\right] \\
&= \frac{1}{2}\frac{\left(1-\frac{\rho^4}{b^4}+2i\frac{\rho^2}{b^2}\sin(2\varphi)\right)+\left(1-\frac{\rho^4}{b^4}-2i\frac{\rho^2}{b^2}\sin(2\varphi)\right)}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)} \\
&= \frac{1}{2}\frac{2-2\frac{\rho^4}{b^4}}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)} \\
&= \frac{1-\frac{\rho^4}{b^4}}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)}
\end{aligned}$$

$$\begin{aligned}
\text{Imag}[R] &= \frac{1}{2i}\left[\left(\frac{1+\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}{1-\left(\frac{\rho^2}{b^2}e^{i2\varphi}\right)}\right)-\left(\frac{1+\left(\frac{\rho^2}{b^2}e^{-i2\varphi}\right)}{1-\left(\frac{\rho^2}{b^2}e^{-i2\varphi}\right)}\right)\right] \\
&= \frac{1}{2i}\frac{\left(1-\frac{\rho^4}{b^4}+2i\frac{\rho^2}{b^2}\sin(2\varphi)\right)-\left(1-\frac{\rho^4}{b^4}-2i\frac{\rho^2}{b^2}\sin(2\varphi)\right)}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)} \\
&= \frac{1}{2i}\frac{4i\frac{\rho^2}{b^2}\sin(2\varphi)}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)} \\
&= \frac{2\frac{\rho^2}{b^2}\sin(2\varphi)}{1+\frac{\rho^4}{b^4}-2\frac{\rho^2}{b^2}\cos(2\varphi)}
\end{aligned}$$

$$
\begin{aligned}
\Phi(\rho,\varphi) &= \frac{4V}{\pi}\tan^{-1}\left(\frac{\text{Imag}[R]}{\text{Real}[R]}\right)\\
&= \frac{4V}{\pi}\tan^{-1}\left(\frac{2\frac{\rho^2}{b^2}\sin(2\varphi)}{1-\frac{\rho^4}{b^4}}\right)\\
&= \boxed{\frac{4V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2\sin(2\varphi)}{b^4-\rho^4}\right)}
\end{aligned}
$$

2.14.c. Sketch the field lines and equipotentials.

Problem 2.23

A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

2.23.a. Find the potential $\Phi(x, y, z)$ at any point inside the cube.

$$
\begin{aligned}
\nabla^2\Phi(x,y,z) &= 0\\
\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} &= 0
\end{aligned}
$$

Assume $\Phi(x, y, z) = X(x)Y(y)Z(z)$:

$$X''YZ + XY''Z + XYZ'' = 0$$

Dividing both sides by XYZ :

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\begin{aligned}X''&=\alpha^2X\\Y''&=\beta^2Y\\Z''&=\left(\alpha^2+\beta^2\right)X\end{aligned}$$

$$\begin{aligned}X(x)&=A\sin(\alpha x)+B\cos(\alpha x)\\Y(y)&=C\sin(\beta x)+D\cos(\beta x)\\Z(z)&=Ee^{-z\sqrt{\alpha^2+\beta^2}}+Fe^{z\sqrt{\alpha^2+\beta^2}}\end{aligned}$$

$$\begin{aligned}X(0)&=B=0\\X(a)&=A\sin(\alpha a)=0\implies\alpha=\frac{n\pi}{a}\\Y(0)&=D=0\\Y(a)&=C\sin(\alpha a)=0\implies\beta=\frac{m\pi}{a}\end{aligned}$$

$$\Phi(x,y,z)=\sum_{m,n}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\left[A_1e^{-z\frac{\pi}{a}\sqrt{n^2+m^2}}+B_1e^{z\frac{\pi}{a}\sqrt{n^2+m^2}}\right]$$

Using $\Phi(x,y,z=0)=V$:

$$\begin{aligned}A_1+B_1&=\frac{2}{a}\frac{2}{a}\int_{y=0}^{a}\int_{x=0}^{a}(V)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)dxdy\\A_1+B_1&=\frac{4}{a^2}V\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]_0^a\left[-\frac{a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)\right]_0^a\\A_1+B_1&=\frac{4V}{a^2}\frac{4a^2}{mn\pi^2}\\A_1+B_1&=\frac{16V}{mn\pi^2}\qquad(1)\end{aligned}$$

Using $\Phi(x,y,z=a)=V$:

$$\begin{aligned}A_1e^{-a\frac{\pi}{a}\sqrt{n^2+m^2}}+B_1e^{a\frac{\pi}{a}\sqrt{n^2+m^2}}&=\frac{2}{a}\frac{2}{a}\int_{y=0}^{a}\int_{x=0}^{a}(V)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)dxdy\\A_1e^{-\pi\sqrt{n^2+m^2}}+B_1e^{\pi\sqrt{n^2+m^2}}&=\frac{16V}{mn\pi^2}\qquad(2)\end{aligned}$$

Solving equation (1) for B_1 and plugging it into equation (2):

$$A_1e^{-\pi\sqrt{n^2+m^2}}+\left(\frac{16V}{mn\pi^2}-A_1\right)e^{\pi\sqrt{n^2+m^2}}=\frac{16V}{mn\pi^2}$$

$$A_1 = \frac{16V}{nm\pi^2}\frac{e^{\pi\sqrt{n^2+n^2}} - 1}{e^{\pi\sqrt{n^2+m^2}} - e^{-\pi\sqrt{n^2+m^2}}}$$

$$A_1 = \frac{16V}{nm\pi^2}\frac{e^{\pi\sqrt{n^2+n^2}} - 1}{\left(e^{\frac{\pi}{2}\sqrt{n^2+m^2}} - e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}\right)\left(e^{\frac{\pi}{2}\sqrt{n^2+m^2}} + e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}\right)} e^{-\frac{\pi}{2}\sqrt{n^2+m^2}} e^{\frac{\pi}{2}\sqrt{n^2+m^2}}$$

$$A_1 = \frac{16V}{nm\pi^2}\frac{e^{\frac{\pi}{2}\sqrt{n^2+n^2}} - e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}}{\left(e^{\frac{\pi}{2}\sqrt{n^2+m^2}} - e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}\right)\left(e^{\frac{\pi}{2}\sqrt{n^2+m^2}} + e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}\right)} e^{\frac{\pi}{2}\sqrt{n^2+m^2}}$$

$$A_1 = \frac{16V}{nm\pi^2}\frac{e^{\frac{\pi}{2}\sqrt{n^2+m^2}}}{e^{\frac{\pi}{2}\sqrt{n^2+m^2}} + e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}}$$

$$A_1 = \frac{8V}{nm\pi^2}\frac{e^{\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

Plugging this into equation (1):

$$B_1 = \frac{16}{mn\pi^2} - \frac{8V}{nm\pi^2}\frac{e^{\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$= \frac{8V}{nm\pi^2}\frac{\left(e^{\frac{\pi}{2}\sqrt{n^2+m^2}} + e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}\right) - e^{\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$= \frac{8V}{nm\pi^2}\frac{e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$\Phi(x,y,z) = \sum_{m,n} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\left[\left(\frac{8V}{nm\pi^2}\frac{e^{\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right)e^{-z\frac{\pi}{a}\sqrt{n^2+m^2}} + \left(\frac{8V}{nm\pi^2}\frac{e^{-\frac{\pi}{2}\sqrt{n^2+m^2}}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right)e^{z\frac{\pi}{a}\sqrt{n^2+m^2}}\right]$$

$$= \frac{8V}{\pi^2}\sum_{m,n}\frac{1}{nm}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\left(\frac{e^{\pi\sqrt{n^2+m^2}\left(\frac{1}{2}-\frac{z}{a}\right)} + e^{-\pi\sqrt{n^2+m^2}\left(\frac{1}{2}-\frac{z}{a}\right)}}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right)$$

$$\boxed{\Phi(x,y,z) = \frac{16V}{\pi^2}\sum_{m,n}\frac{1}{nm}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\left(\frac{\cosh\left[\pi\sqrt{n^2+m^2}\left(\frac{1}{2}-\frac{z}{a}\right)\right]}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right)}$$

2.23.b.

Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential of the walls. See problem 2.28.

The following Matlab code was used:

```
for T = 1:10
  Phi = 0;
  for n=1:T
    for m=1:T
      Phi = Phi + sin(n*pi/2)*sin(m*pi/2)/cosh((pi/2)*sqrt(n^2+m^2));
    end
  end
  Phi*16/(pi^2)
end
```

Note that the `cosh` term in the numerator is exactly 1 at $z = a/2$. The upper limit of the summation of both m and n was increased. The results are shown in the table below:

T	Φ/V
1	0.34755
2	0.34755
3	0.30654
4	0.30654
5	0.30806
6	0.30806
7	0.30800
8	0.30800
9	0.30800
10	0.30800

Note that three decimal places of accuracy is reached at $T = 5$, where both summations in m and n are evaluated from 1 to T (i.e., 25 terms in total).

The average potential over the six faces of the cube is:

$$\frac{V+V}{6} = \frac{V}{3}$$

This is close to the $\Phi = 0.308V$ we found at the center of the cube.

2.23.c. Find the surface-charge density on the surface $z = a$.

$$\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=a} \\
&= -\varepsilon_0 \frac{16V}{\pi^2} \sum_{m,n} \frac{1}{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \left(\frac{\sinh\left[\pi\sqrt{n^2+m^2}\left(\frac{1}{2}-\frac{z}{a}\right)\right]}{\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\right) \left(-\frac{\pi}{a}\sqrt{n^2+m^2}\right)\bigg|_{z=a} \\
&= \boxed{-\varepsilon_0 \frac{16V}{a\pi} \sum_{m,n} \frac{1}{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) \tanh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right) \sqrt{n^2+m^2}}
\end{aligned}$$