Problem 2.7

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

2.7.a. Write down the appropriate Green function $G(\vec{x}, \vec{x}')$.

$$G_D(\vec{x}, \vec{x}') = \frac{1}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}} - \frac{1}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 + x_3')^2}}$$

where x_1, x_2, and x_3 denote the x, y , and z coordinates, respectively.

2.7.b.

If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, φ, z).

We're going to use equation 1.44 from Jackson:

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} da' \tag{1}$$

Note that $\Phi(\vec{x}') = V$ inside the circle of radius a centered at the origin.

Let's convert $G_D(\vec{x}, \vec{x}')$ to cylindrical coordinates:

$$\begin{aligned}
G_D(\vec{x}, \vec{x}') =& \frac{1}{\sqrt{(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2 + (z - z')^2}} \\
&- \frac{1}{\sqrt{(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2 + (z + z')^2}} \\
=& \frac{1}{\sqrt{\rho^2 + \rho'^2 - \rho\rho'(\underbrace{\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'}_{\cos(\varphi - \varphi')}) + (z - z')^2}} \\
&- \frac{1}{\sqrt{\rho^2 + \rho'^2 - \rho\rho'(\underbrace{\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'}_{\cos(\varphi - \varphi')}) + (z + z')^2}} \\
=& \frac{1}{\sqrt{\rho^2 + \rho'^2 - \rho\rho'\cos(\varphi - \varphi') + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - \rho\rho'\cos(\varphi - \varphi') + (z + z')^2}}
\end{aligned}$$

We need to find the normal derivative of G_D. Note that the normal points away from the region of interest – since we're considering $z \geq 0$, let $\hat{n} = -\hat{z}'$:

$$\begin{aligned}
\frac{\partial G_D}{\partial n'} &= \nabla G_D(\vec{x}, \vec{x}') \cdot (-\hat{z}') \\
&= -\left[\cancelto{0}{\frac{1}{\rho'}\frac{\partial}{\partial \rho'}(\rho' G_D)} + \cancelto{0}{\frac{1}{\rho'}\frac{\partial G_D}{\partial \varphi'}} + \frac{\partial G_D}{\partial z'}\bigg|_{z'=0}\right] \\
&= -\left[-\frac{1}{2}\frac{2(z-z')(-1)}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+(z-z')^2)^{3/2}} + \frac{1}{2}\frac{2(z+z')}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+(z+z')^2)^{3/2}}\right]_{z'=0} \\
&= -\frac{z}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}} - \frac{z}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}} \\
&= -\frac{2z}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}}
\end{aligned}$$

Note that the terms which "cancel to zero" do so because the derivatives of the two terms of G_D sum to zero when evaluated at $z' = 0$.

Now, plug this into equation (1). Note that we only need to integrate over the circle which has potential V because the integrand is zero elsewhere.

$$\begin{aligned}
\Phi(\vec{x}) &= -\frac{1}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}(V)\left(-\frac{2z}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}}\right)\rho' d\rho' d\varphi' \\
&= \boxed{\frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\frac{\rho'}{(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}} d\rho' d\varphi'}
\end{aligned}$$

2.7.c. Show that, along the axis of the circle ($\rho = 0$), the potential is $\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$.

Letting $\rho = 0$:

$$\Phi(\vec{x}) = \frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\frac{\rho'}{(\rho'^2+z^2)^{3/2}} d\rho' d\varphi'$$

Using the substitution $u = \rho'^2 + z^2$ and $du = 2\rho'$:

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\left[\int_{u=z^2}^{a^2+z^2}\frac{\frac{1}{2}du}{u^{3/2}}\right]d\varphi' \\
&= \frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\left[(-2)\frac{\frac{1}{2}}{u^{1/2}}\right]_{u=z^2}^{a^2+z^2}d\varphi' \\
&= -\frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\left[\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{\sqrt{z^2}}\right]d\varphi' \\
&= -\frac{Vz}{2\pi}2\pi\left[\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{z}\right] \\
&= \boxed{V\left[1-\frac{z}{\sqrt{a^2+z^2}}\right]}
\end{aligned}$$

2.7.d.

Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are:

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2}+\dots\right]$$

Verify that the results of part c and d are consistent with each other in their common range of validity.

$$\Phi(\vec{x}) = \frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\frac{\rho'}{\left(\rho^2+\rho'^2-\rho\rho'\cos(\varphi-\varphi')+z^2\right)^{3/2}}d\rho' d\varphi'$$

$$=\frac{Vz}{2\pi}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\rho'\left(\rho^2+z^2\right)^{-3/2}\left(1+\frac{\rho'^2-\rho\rho'\cos(\varphi-\varphi')}{\rho^2+z^2}\right)^{-3/2}d\rho' d\varphi'$$

Using the Taylor expansion $(1+x)^n = 1+nx+n(n-1)x^2+\ldots$

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\rho'\left(1-\frac{3}{2}\frac{\rho'^2-\rho\rho'\cos(\varphi-\varphi')}{\rho^2+z^2}+\frac{15}{4}\left(\frac{\rho'^2-\rho\rho'\cos(\varphi-\varphi')}{\rho^2+z^2}\right)^2+\ldots\right)d\rho' d\varphi'$$

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\rho'-\frac{3}{2}\frac{\rho'^3-\rho\rho'^2\cos(\varphi-\varphi')}{\rho^2+z^2}+\frac{15}{4}\left(\frac{\rho'^2-\rho\rho'\cos(\varphi-\varphi')}{\rho^2+z^2}\right)^2+\ldots d\rho' d\varphi'$$

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_{\varphi'=0}^{2\pi}\int_{\rho'=0}^{a}\rho'-\frac{3}{2}\frac{\rho'^3-\rho\rho'^2\cos(\varphi-\varphi')}{\rho^2+z^2}$$
$$+\frac{15}{4}\frac{\rho'^5+\rho^2\rho'^3\cos^2(\varphi-\varphi')-2\rho\rho'^4\cos(\varphi-\varphi')}{\left(\rho^2+z^2\right)^2}+\ldots d\rho' d\varphi'$$

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_{\varphi'=0}^{2\pi}\frac{1}{2}\rho'^2-\frac{3}{2}\frac{\frac{1}{4}\rho'^4-\rho\frac{1}{3}\rho'^3\cos(\varphi-\varphi')}{\rho^2+z^2}$$
$$+\frac{15}{4}\frac{\frac{1}{6}\rho'^6+\rho^2\frac{1}{4}\rho'^4\cos^2(\varphi-\varphi')-2\rho\frac{1}{5}\rho'^5\cos(\varphi-\varphi')}{\left(\rho^2+z^2\right)^2}+\ldots\Big|_{\rho'=0}^{a}d\varphi'$$

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\int_{\varphi'=0}^{2\pi}\frac{1}{2}a^2-\frac{3}{2}\frac{\frac{1}{4}a^4-\rho\frac{1}{3}a^3\cancelto{0}{\cos(\varphi-\varphi')}}{\rho^2+z^2}$$
$$+\frac{15}{4}\frac{\frac{1}{6}a^6+\rho^2\frac{1}{4}a^4\cos^2(\varphi-\varphi')-2\rho\frac{1}{5}a^5\cancelto{0}{\cos(\varphi-\varphi')}}{\left(\rho^2+z^2\right)^2}+\ldots d\varphi'$$

Note that the "canceled" terms integrate to zero.

$$=\frac{Vz}{2\pi\left(\rho^2+z^2\right)^{3/2}}\left(2\pi\frac{1}{2}a^2-2\pi\frac{3}{2}\frac{\frac{1}{4}a^4}{\rho^2+z^2}+\frac{15}{4}\frac{2\pi\frac{1}{6}a^6+\pi\rho^2\frac{1}{4}a^4}{\left(\rho^2+z^2\right)^2}+\ldots\right)$$

$$=\boxed{\frac{Vza^2}{2\left(\rho^2+z^2\right)^{3/2}}\left(1-\frac{3a^2}{4\left(\rho^2+z^2\right)}+\frac{5\left(a^4+3\rho^2a^2\right)}{4\left(\rho^2+z^2\right)^2}+\ldots\right)}$$

For $\rho = 0$:

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{Va^2}{2z^2}\left(1 - \frac{3a^2}{4z^2} + \frac{5a^4}{4z^4} + \dots\right) \\
&= V\left(\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \frac{5a^6}{8z^6} + \dots\right) \\
&= V\left[1 - \underbrace{\left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} - \frac{5a^6}{8z^6} + \dots\right)}_{\left(1+\frac{a^2}{z^2}\right)^{-1/2}}\right] \\
&= V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]
\end{aligned}$$

Hence, parts c and d agree in the limit where $z^2 \gg a^2$.

Problem 2.9

An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

2.9.a. If the shell is uncharged.

From Jackson's example problem in section 2.5, we know that the surface-charge density is given by:

$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

Using the equation shown in figure 2.4:

$$\begin{aligned}
\frac{dF}{da} &= \frac{\sigma^2}{2\varepsilon_0} \\
&= \frac{9\varepsilon_0^2 E_0^2 \cos^2\theta}{2\varepsilon_0} \\
\frac{dF_z}{da} &= \frac{9\varepsilon_0^2 E_0^2 \cos^2\theta}{2\varepsilon_0}\cos\theta
\end{aligned}$$

$$|F_z| = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{9}{2}\varepsilon_0 E_0^2 \cos^3\theta \rho^2 \sin\theta d\theta d\varphi\Big|_{\rho=a}$$

Using the substitution $u = \cos\theta$, $du = \sin\theta$:

$$\begin{aligned}
|F_z| &= \int_{\varphi=0}^{2\pi} \int_{u=1}^{0} \frac{9}{2}\varepsilon_0 E_0^2 u^3 (-du) d\varphi \\
&= -\frac{9}{2} a^2 \varepsilon_0 E_0^2 \int_{\varphi=0}^{2\pi} \int_{u=1}^{0} u^3 du d\varphi \\
&= -\frac{9}{2} \varepsilon_0 a^2 E_0^2 \int_{\varphi=0}^{2\pi} \frac{1}{4} u^4 \Big|_1^0 d\varphi \\
&= \boxed{\frac{9}{4}\pi\varepsilon_0 a^2 e_0^2}
\end{aligned}$$

2.9.b. If the total charge on the shell is Q.

The surface charge density on the sphere of charge Q is:

$$\sigma_Q = \frac{Q}{4\pi a^2}$$

Now, we find the same method as above to find the force between the two hemispheres of equal charge:

$$\begin{aligned}
\frac{dF}{da} &= \frac{\sigma^2}{2\varepsilon_0} \\
&= \frac{Q^2}{16\pi^2 a^4} \frac{1}{2\varepsilon_0} \\
\frac{dF_z}{da} &= \frac{Q^2}{32\pi^2 \varepsilon_0 a^4} \cos\theta
\end{aligned}$$

$$|F_z| = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q^2}{32\pi^2\varepsilon_0 a^4} \cos\theta \rho^2 \sin\theta d\theta d\varphi \Big|_{\rho=a}$$

Using the substitution $u = \sin\theta$, $du = \cos\theta$:

$$\begin{aligned}
|F_z| &= \frac{Q^2}{32\pi^2\varepsilon_0 a^2} \int_{\varphi=0}^{2\pi} \int_{u=0}^{1} u du d\varphi \\
&= \frac{Q^2}{32\pi^2\varepsilon_0 a^2} \int_{\varphi=0}^{2\pi} \frac{1}{2} d\varphi \\
&= \frac{Q^2}{32\pi^2\varepsilon_0 a^2} \frac{1}{2} 2\pi \\
&= \frac{Q^2}{32\pi\varepsilon_0 a^2}
\end{aligned}$$

The total force on the sphere due to the sphere's own charge Q and the electric field is the sum of the force found in part a and the force we just found:

$$\boxed{|F_{total}| = \frac{9}{4}\pi\varepsilon_0 a^2 e_0^2 + \frac{Q^2}{32\pi\varepsilon_0 a^2}}$$

Problem 2.10

A large parallel plate capacitor is made up of two plane conducting sheets with separation D, one of which has a small hemispherical boss of radius a on its inner surface $D \gg a$. The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is E_0.

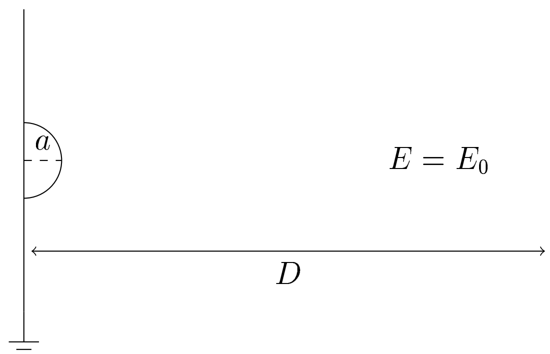

Figure 1: Setup for problem 2.10

2.10.a. Calculate the surface-charge densities at any arbitrary point on the plane and on the boss, and sketch their behavior as a function of distance (or angle).

Assuming the planes to be infinite and very far from each other, we see that this system can be approximated by grounded sphere in a uniform electric field (we are given that the purpose of the non-grounded plate is to cause the electric field between the plates to be constant and uniform). Hence, equation 2.14 from Jackson gives the electric potential between the plates:

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{2}$$

On the boss, the surface-charge density is the same as equation 2.15 from Jackson:

$$\boxed{\sigma = 3\varepsilon_0 E_0 \cos\theta}$$

To find the surface-charge density on the grounded plane (located at $z = 0$), we first convert equation (2) to Cartesian coordinates:

$$\begin{aligned}
\Phi &= -E_0 \underbrace{r\cos\theta}_{z} \left(1 - \frac{a^3}{r^3}\right) \\
&= -E_0 z \left(1 - \left(\frac{a}{r}\right)^3\right)
\end{aligned}$$

$$\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=0} \\
&= \boxed{-E_0 \left(1 - \left(\frac{a}{r}\right)^3\right)}
\end{aligned}$$

2.10.b. Show that the total charge on the boss has the magnitude $3\pi\varepsilon_0 E_0 a^2$.

We will integrate the surface-charge density over the surface area of the boss to find its net charge:

$$\begin{aligned}
Q &= \int 3\varepsilon_0 E_0 \cos\theta da \\
&= 3\varepsilon_0 E_0 \int_{\varphi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \cos\theta \rho^2 \sin\theta d\theta d\varphi\bigg|_{\rho=a} \\
&= 3a^2\varepsilon_0 E_0 \int_{\varphi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \cos\theta \sin\theta d\theta d\varphi
\end{aligned}$$

Using the substitution $u = \sin\theta$, $du = \cos\theta$:

$$\begin{aligned}
Q &= 3a^2\varepsilon_0 E_0 \int_{\varphi=0}^{2\pi} \int_{u=0}^{-1} u du d\varphi \\
&= 3a^2\varepsilon_0 E_0 \int_{\varphi=0}^{2\pi} \frac{1}{2} u^2 \bigg|_{u=0}^{-1} d\varphi \\
&= 3a^2\varepsilon_0 E_0 \frac{1}{2} 2\pi \\
&= \boxed{3\pi a^2 \varepsilon_0 E_0}
\end{aligned}$$

2.10.c.

If, instead of the other conducting sheet at a different potential, a point charge q is placed directly above the hemispherical boss at a distance d from its center, show that the charge induced on the boss is:

$$q' = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right]$$

This system is shown in figure 2.

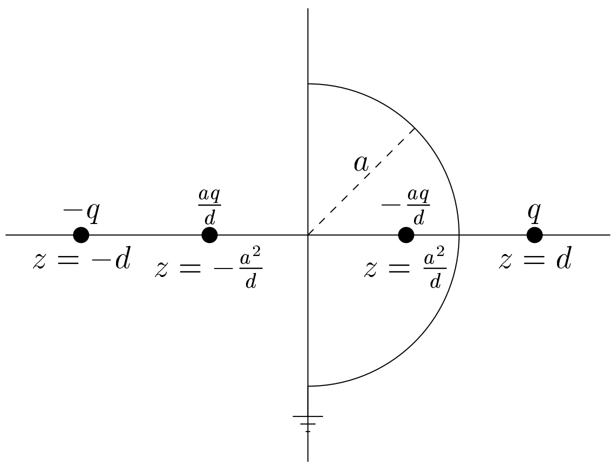

Figure 2: Setup for problem 2.10.c

The potential for this system is:

$$\begin{aligned}\Phi =& \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x_1^2 + x_2^2 + (x_3 - d)^2}} + \frac{-\frac{aq}{d}}{\sqrt{x_1^2 + x_2^2 + \left(x_3 - \frac{a^2}{d}\right)^2}}\right. \\ &\left. + \frac{\frac{aq}{d}}{\sqrt{x_1^2 + x_2^2 + \left(x_3 + \frac{a^2}{d}\right)^2}} + \frac{-q}{\sqrt{x_1^2 + x_2^2 + (x_3 + d)^2}}\right]\end{aligned}$$

Converting to spherical coordinates:

$$\begin{aligned}\Phi =& \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{\rho^2 + 2d\rho\cos\theta + d^2}} + \frac{-\frac{aq}{d}}{\sqrt{\rho^2 + 2\frac{a^2}{d}\rho\cos\theta + \left(\frac{a^2}{d}\right)^2}}\right. \\ &\left. + \frac{\frac{aq}{d}}{\sqrt{\rho^2 - 2\frac{a^2}{d}\rho\cos\theta + \left(\frac{a^2}{d}\right)^2}} + \frac{-q}{\sqrt{\rho^2 - 2d\rho\cos\theta + d^2}}\right]\end{aligned}$$

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{\rho=a} = -\frac{1}{4\pi}\left[-\frac{1}{2}\frac{q\left(2\rho+2d\cos\theta\right)}{\left(\rho^2+2d\rho\cos\theta+d^2\right)^{3/2}} - \frac{1}{2}\frac{-\frac{aq}{d}\left(2\rho+2\frac{a^2}{d}\cos\theta\right)}{\left(\rho^2+2\frac{a^2}{d}\rho\cos\theta+\left(\frac{a^2}{d}\right)^2\right)^{3/2}}\right.$$
$$\left.-\frac{1}{2}\frac{\frac{aq}{d}\left(2\rho-2\frac{a^2}{d}\cos\theta\right)}{\left(\rho^2-2\frac{a^2}{d}\rho\cos\theta+\left(\frac{a^2}{d}\right)^2\right)^{3/2}} - \frac{1}{2}\frac{-q\left(2\rho-2d\cos\theta\right)}{\left(\rho^2-2d\rho\cos\theta+d^2\right)^{3/2}}\right]$$

$$\begin{aligned}
Q &= \int_{\varphi=0}^{2\pi}\int_{\theta=\pi/2}^{\pi}\left(\sigma\right)\rho^2\sin\theta d\theta d\varphi\Big|_{\rho=a} \\
&= \frac{1}{8\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=\pi/2}^{\pi}\left[\frac{q\left(2a+2d\cos\theta\right)}{\left(a^2+2da\cos\theta+d^2\right)^{3/2}} + \frac{-\frac{aq}{d}\left(2a+2\frac{a^2}{d}\cos\theta\right)}{\left(a^2+2\frac{a^2}{d}a\cos\theta+\left(\frac{a^2}{d}\right)^2\right)^{3/2}}\right. \\
&\left.+\frac{\frac{aq}{d}\left(2a-2\frac{a^2}{d}\cos\theta\right)}{\left(a^2-2\frac{a^2}{d}a\cos\theta+\left(\frac{a^2}{d}\right)^2\right)^{3/2}} + \frac{-q\left(2a-2d\cos\theta\right)}{\left(a^2-2da\cos\theta+d^2\right)^{3/2}}\right]a^2\sin\theta d\theta d\varphi \\
&= \frac{1}{8\pi}\int_{\varphi=0}^{2\pi}\left[-2q\frac{\sqrt{a^2+d^2}-d}{\sqrt{a^2+d^2}} + 2qa\frac{-\sqrt{a^2+d^2}-a}{d\sqrt{a^2+d^2}} + 2qa\frac{\sqrt{a^2+d^2}-a}{d\sqrt{a^2+d^2}} - 2q\frac{\sqrt{a^2+d^2}-d}{\sqrt{a^2+d^2}}\right]d\varphi \\
&= \frac{1}{8\pi}2\pi\left[-2q\frac{\sqrt{a^2+d^2}-d}{\sqrt{a^2+d^2}} + 2qa\frac{-\sqrt{a^2+d^2}-a}{d\sqrt{a^2+d^2}} + 2qa\frac{\sqrt{a^2+d^2}-a}{d\sqrt{a^2+d^2}} - 2q\frac{\sqrt{a^2+d^2}-d}{\sqrt{a^2+d^2}}\right] \\
&= -q\frac{\sqrt{a^2+d^2}-d}{\sqrt{a^2+d^2}} - q\frac{a^2}{d\sqrt{a^2+d^2}} \\
&= -q\frac{d\sqrt{a^2+d^2}-\left(d^2-a^2\right)}{d\sqrt{a^2+d^2}} \\
&= -q\left[1-\frac{d^2-a^2}{d\sqrt{a^2+d^2}}\right]
\end{aligned}$$