# Problem 1.14

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface $S$ bounding the volume $V$. Apply Green's theorem (1.35) with integration variable $\vec{y}$ and $\varphi = G(\vec{x}, \vec{y})$, $\psi = G(\vec{x}', \vec{y})$, with $\nabla_y^2 G(\vec{z}, \vec{y}) = -4\pi\delta(\vec{y} - \vec{z})$. Find an expression for the difference $[G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x})]$ in terms of an integral over the boundary surface $S$.

Starting with equation 1.35:

$$
\begin{aligned}
\int_V \left(\varphi \nabla^2 \psi - \psi \nabla^2 \varphi\right) d^3y &= \oint_S \left[\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right] da \\
\int_V \left(G(\vec{x}, \vec{y}) \nabla_y^2 G(\vec{x}', \vec{y}) - G(\vec{x}', \vec{y}) \nabla_y^2 G(\vec{x}, \vec{y})\right) d^3y &= \oint_S \left[\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right] da \\
\int_V G(\vec{x}, \vec{y}) \left[-4\pi\delta(\vec{y} - \vec{x}')\right] d^3y - \int_V G(\vec{x}', \vec{y}) \left[-4\pi\delta(\vec{y} - \vec{x})\right] d^3y &= \oint_S \left[\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right] da \\
-4\pi G(\vec{x}, \vec{x}') + 4\pi G(\vec{x}', \vec{x}) &= \oint_S \left[\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right] da
\end{aligned}
$$

$$
\boxed{G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x}) = -\frac{1}{4\pi} \oint_S \left[G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n}\right] da}
$$

## 1.14.a

For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\vec{x}, \vec{x}')$ must be symmetric in $\vec{x}$ and $\vec{x}'$.

Note that $G_D(\vec{x}, \vec{y}) = 0$ and $G_D(\vec{x}', \vec{y}) = 0$ on the bounding surface. Therefore:

$$
\begin{aligned}
&G_D(\vec{x}, \vec{x}') - G_D(\vec{x}', \vec{x}) = -\frac{1}{4\pi} \oint_S \left[\cancelto{0}{G_D(\vec{x}, \vec{y})} \frac{\partial G_D(\vec{x}', \vec{y})}{\partial n} - \cancelto{0}{G_D(\vec{x}', \vec{y})} \frac{\partial G_D(\vec{x}, \vec{y})}{\partial n}\right] da \\
&G_D(\vec{x}, \vec{x}') - G_D(\vec{x}', \vec{x}) = 0
\end{aligned}
$$

$$
\therefore \boxed{G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})}
$$

## 1.14.b

For Neumann boundary conditions, use the boundary conditions (1.45) for $G_N(\vec{x},\vec{x}')$ to show that $G_N(\vec{x},\vec{x}')$ is not symmetric in general but that $G_N(\vec{x},\vec{x}') - F(\vec{x})$ is symmetric in $\vec{x}$ and $\vec{x}'$, where

$$F(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{x}')da_y$$

$$G_N(\vec{x},\vec{x}') - G_N(\vec{x}',\vec{x}) = -\frac{1}{4\pi}\oint_S \left[G_N(\vec{x},\vec{y})\frac{\partial G_N(\vec{x}',\vec{y})}{\partial n} - G_N(\vec{x}',\vec{y})\frac{\partial G_N(\vec{x},\vec{y})}{\partial n}\right] da$$

$$G_N(\vec{x},\vec{x}') - G_N(\vec{x}',\vec{x}) = -\frac{1}{4\pi}\oint_S \left[G_N(\vec{x},\vec{y})\left(-\frac{4\pi}{S}\right) - G_N(\vec{x}',\vec{y})\left(-\frac{4\pi}{S}\right)\right] da$$

$$G_N(\vec{x},\vec{x}') - G_N(\vec{x}',\vec{x}) = \frac{1}{S}\oint_S \left[G_N(\vec{x},\vec{y}) - G_N(\vec{x}',\vec{y})\right] da$$

$$G_N(\vec{x},\vec{x}') - \underbrace{\frac{1}{S}\oint_S G_N(\vec{x},\vec{y})da}_{F(\vec{x})} = G_N(\vec{x}',\vec{x}) - \underbrace{\frac{1}{S}\oint_S G_N(\vec{x}',\vec{y})da}_{F(\vec{x}')}$$

$$\therefore \boxed{G_N(\vec{x},\vec{x}') - F(\vec{x}) = G_N(\vec{x}',\vec{x}) - F(\vec{x}')}$$

## 1.14.c

Show that the addition of $F(\vec{x})$ to the Green function does not affect the potential $\Phi(\vec{x})$. See problem 3.26 for an example of the Neumann Green function.

Note that $F(\vec{x})$ is independent of $\vec{x}'$. Therefore:

$$\nabla'^2 F(\vec{x}) = 0$$
$$\frac{\partial F(\vec{x})}{\partial n'} = 0$$

So:

$$\nabla'^2 G(\vec{x},\vec{x}') = \nabla'^2\left[G_N(\vec{x},\vec{x}') + F(\vec{x})\right] = -4\pi\delta(\vec{x}-\vec{x}')$$
$$\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} = \frac{\partial}{\partial n'}\left[G_N(\vec{x},\vec{x}') + F(\vec{x})\right] = -\frac{4\pi}{S}$$

Therefore, the addition of $F(\vec{x})$ does not affect the fact that $G(\vec{x},\vec{x}')$ is a valid Green function. As a result, the fact that $\Phi(\vec{x})$ is a solution remains unaffected. Because $\Phi(\vec{x})$ is a unique solution, the fact that $\Phi(\vec{x})$ remains a solution must mean that $\Phi(\vec{x})$ remains unaffected.

# Problem 2.1

A point charge $q$ is brought to a position a distance $d$ away from an infinite plane conductor held at zero potential. Using the method of images, find:

## 2.1.a. The surface-charge density induced on the plane, and plot it

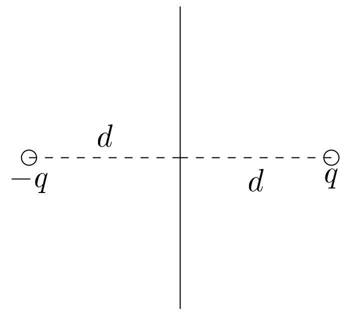


Figure 1:

$$
\begin{aligned}
\Phi &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{d}|} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{|\vec{x}+\vec{d}|} \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x_1-d_1)^2+x_2^2+x_3^2}} - \frac{1}{\sqrt{(x_1+d_1)^2+x_2^2+x_3^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial}{\partial x_1}\Phi\bigg|_{x_1=0} \\
&= -\frac{q}{4\pi}\left(\frac{\left(-\frac{1}{2}\right)2(x_1-d_1)}{[(x_1-d_1)^2+x_2^2+x_3^2]^{3/2}} - \frac{\left(-\frac{1}{2}\right)2(x_1+d_1)}{[(x_1+d_1)^2+x_2^2+x_3^2]^{3/2}}\right)\bigg|_{x_1=0} \\
&= -\frac{q}{4\pi}\left(\frac{d_1}{[d_1^2+x_2^2+x_3^2]^{3/2}} + \frac{d_1}{[d_1^2+x_2^2+x_3^2]^{3/2}}\right) \\
&= \boxed{-\frac{q}{2\pi}\frac{d_1}{[d_1^2+x_2^2+x_3^2]^{3/2}}}
\end{aligned}
$$

$\sigma$ is plotted in figure 2 with $q=1$ and $x_2, x_3 = 0$.

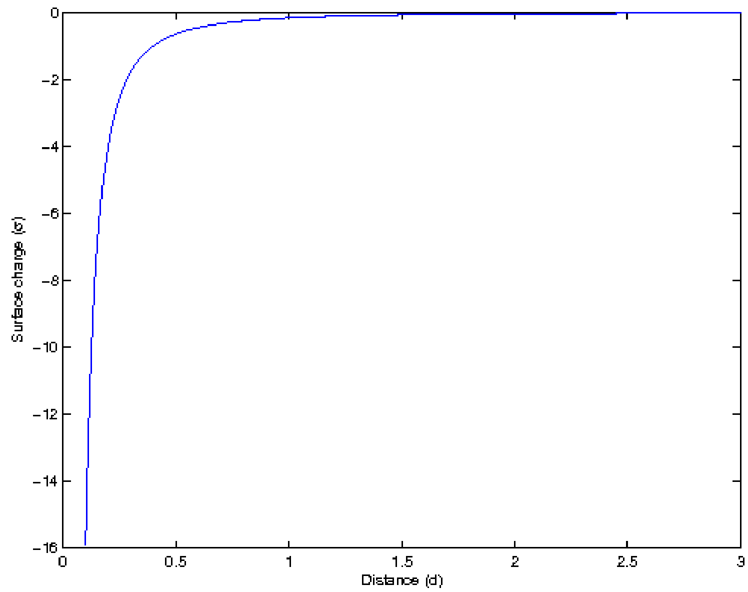


Figure 2: Plot for problem 2.1.a.

**2.1.b. The force between the plane and the charge by using Coulomb's law of the force between the charge and its image**

$$F = -\frac{q(-q)}{4\pi\varepsilon_0(2d)^2}$$

$$\boxed{F = \frac{q^2}{16\pi\varepsilon_0 d^2}}$$

**2.1.c. The total force acting on the plane by integrating $\sigma^2/2\varepsilon_0$ over the whole plane**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma^2}{2\varepsilon_0}dx_2dx_3 = -\frac{q^2d_1^2}{8\pi^2\varepsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{[d_1^2+x_2^2+x_3^2]^3}dx_2dx_3$$

Convert to polar coordinates:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma^2}{2\varepsilon_0}dx_2dx_3 &= -\frac{q^2d_1^2}{8\pi^2\varepsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{1}{[d_1^2+\rho^2]^3}\rho d\rho d\theta \\
&= -\frac{q^2d_1^2}{8\pi^2\varepsilon_0}\int_0^{2\pi}\int_{d_1^2}^{\infty}\frac{1}{u^3}\left[\frac{1}{2}du\right]d\theta; \quad u = d_1^2+\rho^2 \\
&= -\frac{q^2d_1^2}{16\pi^2\varepsilon_0}\int_0^{2\pi}\left[-\frac{1}{2}u^{-2}\right]_{d_1^2}^{\infty}d\theta \\
&= \frac{q^2d_1^2}{32\pi^2\varepsilon_0 d_1^4}\int_0^{2\pi}d\theta \\
&= \frac{q^2}{32\pi^2\varepsilon_0 d_1^2}[2\pi] \\
&= \boxed{\frac{q^2}{16\pi\varepsilon_0 d_1^2}}
\end{aligned}$$

## 2.1.d. The work necessary to remove the charge $q$ from its position to infinity

$$\begin{aligned}
W &= \int_d^{\infty} F\cdot dx \\
&= \int_d^{\infty}\frac{q^2}{16\pi\varepsilon_0 x^2}dx \\
&= \left[-\frac{q^2}{16\pi\varepsilon_0 x}\right]_d^{\infty} \\
&= \boxed{\frac{q^2}{16\pi\varepsilon_0 d}}
\end{aligned}$$

### 2.1.e. The potential energy between the charge $q$ and its image (compare the answer to part d and discuss)

$$\begin{aligned}
W &= \int_{-d}^{d} \frac{q^2}{16\pi\varepsilon_0 x^2} dx \\
&= \int_{-d}^{d} \frac{q^2}{16\pi\varepsilon_0 x^2} dx \\
&= \left[ -\frac{q^2}{16\pi\varepsilon_0 x} \right]_{-d}^{d} \\
&= \boxed{-\frac{q^2}{8\pi\varepsilon_0 d}}
\end{aligned}$$

Part b gives the work needed to move the charge to $\infty$ and the image charge to $-\infty$. Part d gives the work needed for the charge to swap places with its image charge. We would expect the work needed to swap the charge and image charge to be considerably larger than the work needed to move the charge and image charge to $\pm\infty$ because more work is required when the charges are closer together.

### 2.1.f. Find the answer to part d in electron volts for an electron originally one angstrom from the surface

$$\begin{aligned}
W &= \frac{q^2}{16\pi\varepsilon_0 d} \\
&= \frac{e(1.6 \times 10^{-16} C)}{16\pi\varepsilon_0 (1 \times 10^{-10})} \\
&= 3.6 \text{ KeV}
\end{aligned}$$

# 1 Problem 2.2

Using the method of images, discuss the problem of a point charge $q$ inside a hollow, grounded, conducting sphere of inner radius $a$. Find:

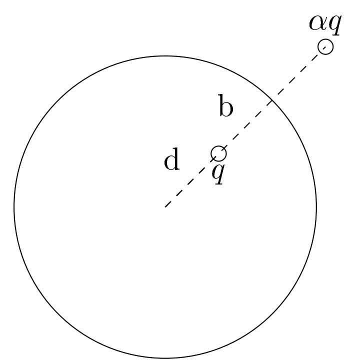


Figure 3:

## 2.2.a. The potential inside the sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{d}|} + \frac{1}{4\pi\varepsilon_0}\frac{\alpha q}{|\vec{x}-\vec{b}|}$$

$$\Phi(x=a) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+d^2-2ad\cos\gamma}} + \frac{\alpha}{\sqrt{a^2+b^2-2ab\cos\gamma}}\right] = 0; \quad \text{where } \vec{x}\cdot\vec{d} = xd\cos\gamma$$

$$\alpha^2(a^2+d^2-2ad\cos\gamma) = a^2+b^2-2ab\cos\gamma$$
$$\alpha^2(a^2+d^2) - 2\alpha^2 ad\cos\gamma = a^2+b^2-2ab\cos\gamma$$

Must be true for all $\gamma$:

$$\alpha^2 d = b$$

$$\alpha^2(a^2+d^2) = a^2+b^2$$
$$\alpha^2(a^2+d^2) = a^2+(\alpha^2 d)^2$$
$$\alpha^4 d^2 - \alpha^2(a^2+d^2) + a^2 = 0$$
$$\alpha^2 = \frac{(a^2+d^2) - \sqrt{(a^2+d^2)^2 - 4a^2d^2}}{2d^2}$$
$$= \frac{(d^2+a^2)-(d^2-a^2)}{2d^2}$$
$$= \frac{2a^2}{2d^2}$$

$$\alpha = -\frac{a}{d}$$
$$b = \frac{a^2}{d}$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\vec{x}-\vec{d}\right|} + \frac{1}{4\pi\varepsilon_0}\frac{-\frac{a}{d}q}{\left|\vec{x}-\frac{a^2}{d^2}\vec{d}\right|}}$$

### 2.2.b. The induced surface-charge density

$$\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial \Phi}{\partial x}\bigg|_{x=a} \\
&= -\varepsilon_0 \frac{\partial}{\partial x}\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+d^2-2xd\cos\gamma}} + \frac{\alpha}{\sqrt{x^2+b^2-2xb\cos\gamma}}\right]\right)\bigg|_{x=a} \\
&= -\frac{q}{4\pi}\left(\frac{2x-2d\cos\gamma}{\left(x^2+d^2-2xd\cos\gamma\right)^{3/2}} + \frac{\alpha(2x-2b\cos\gamma)}{\left(x^2+b^2-2xb\cos\gamma\right)^{3/2}}\right)\bigg|_{x=a} \\
&= \boxed{\frac{q}{4\pi a}\frac{d^2-a^2}{\left(d^2+a^2-2ad\cos\gamma\right)^{3/2}}}
\end{aligned}$$

### 2.2.c. The magnitude and direction of the force acting on $q$

$$\begin{aligned}
F &= -\frac{q(\alpha q)}{4\pi\varepsilon_0(b-d)^2}\hat{d} \\
&= \frac{q^2\frac{a}{d}}{4\pi\varepsilon_0\left(\frac{a^2}{d}-d\right)^2}\hat{d} \\
&= \frac{q^2 ad}{4\pi\varepsilon_0\left(a^2-d^2\right)^2}\hat{d}
\end{aligned}$$

## 2.2.d. Is there any change in the solution if the sphere is kept at a fixed potential $V$? If the sphere has a total charge $Q$ on its inner and outer surfaces?

Yes, there is a change in the solution. How the solution changes for these two cases is detailed below.

#### 1.0.1 Sphere is kept at a fixed potential $V$

Using superposition, we can break this down into two problems. The first is identical to that which we solved above. The second is a single charge at the center of the sphere with a charge such that $\Phi(|\vec{x}| = a) = V$. The final solution is found by solving for the case with all three charges (two charges inside the sphere, one image charge outside).

#### 1.0.2 Sphere has total charge $Q$ on its surfaces

Again, we use superposition to break this down into two problems. And again, the first is identical to that which we solved above. The second is a single charge $Q$ at the center of the sphere. The final solution is found by solving for the case with all three charges.