Problem 1.4

Each of three charged spheres of radius a has a total charge Q. One is conducting, one has a uniform charge density within its volume, and one having a spherically symmetric charge density that varies within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$). Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

1.4.a Conducting sphere

Because charge only resides on the *outside* of a conductor, there is no electric field inside: $\boxed{E = 0}$.

Outside the sphere:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$
$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

where r is the distance from the center from the sphere.

1.4.b Sphere with uniform charge density

Inside the sphere:

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \frac{\cancel{\frac{4}{3}\pi} r^3}{\cancel{\frac{4}{3}\pi} a^3}$$

$$\boxed{E = \frac{Qr}{4\pi a^3}}$$

Outside the sphere, a sphere of charge Q looks the same regardless of its configuration within the sphere:

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

1.4.c Sphere with charge density $\propto r^n$

Inside the sphere:

$$\rho = Cr^n$$

$$\begin{aligned}
Q &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} (Cr^n) r^2 \sin\theta dr d\theta d\varphi \\
&= C \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[r^{n+3}\right]_0^a \sin\theta d\theta d\varphi \\
&= Ca^{n+3} \int_{\varphi=0}^{2\pi} \left[-\cos\theta\right]_0^\pi d\varphi \\
&= C2a^{n+3} \int_{\varphi=0}^{2\pi} d\varphi \\
&= C4\pi a^{n+3}
\end{aligned}$$

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \frac{\cancel{C 4\pi} r^{n+3}}{\cancel{C 4\pi} a^{n+3}}$$

$$\boxed{E = \frac{Q r^{n+1}}{4\pi\varepsilon_0 a^{n+3}}}$$

Outside the sphere:

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

NOTE: This approach is incorrect because $\nabla^2 \left(\frac{1}{r}\right)$ is supposed to be a delta function. See the solutions in "solution.pdf"

$$\begin{aligned}
\vec{E} &= -\nabla\Phi \\
&= -\frac{q}{4\pi\varepsilon_0} \left(\frac{-\alpha r e^{-\alpha r} - e^{-\alpha r}}{r^2} + \frac{\alpha}{2}(-\alpha) e^{-\alpha r} \right) \hat{r} \\
&= \frac{q}{4\pi\varepsilon_0} \left(\frac{\alpha r e^{-\alpha r} + e^{-\alpha r}}{r^2} + \frac{\alpha^2 e^{-\alpha r}}{2} \right) \hat{r}
\end{aligned}$$

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{r^2\left(\alpha e^{-\alpha r} + \alpha r(-\alpha)e^{-\alpha r} - \alpha e^{-\alpha r}\right) - 2r\left(\alpha r e^{-\alpha r} + e^{-\alpha r}\right)}{r^4} - \frac{\alpha^3}{2}e^{-\alpha r}\right) \\
&= -\frac{q}{4\pi\varepsilon_0}e^{-\alpha r}\left(\frac{\alpha^2 r^2 + 2\alpha r + 2}{r^3} + \frac{\alpha^3}{2}\right) = \frac{\rho}{\varepsilon_0}
\end{aligned}
$$

$$
\rho = -\frac{q}{4\pi}e^{-\alpha r}\left(\frac{\alpha^3}{2} + \frac{\alpha^2}{r} + \frac{2\alpha}{r^2} + \frac{2}{r^3}\right)
$$

Problem 1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential difference between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of:

1.6.a Two large, flat conducting sheets of area A, separated by a small distance d

Find the electric field between the plates:

$$
\begin{aligned}
\oint \vec{E} \cdot d\vec{A} &= \frac{Q}{\varepsilon_0} \\
E \cdot A &= \frac{Q}{\varepsilon_0} \\
\left(\frac{V}{d}\right) A &= \frac{Q}{\varepsilon_0} \\
\boxed{C = \varepsilon_0 \frac{A}{d}} &= \frac{Q}{V}
\end{aligned}
$$

1.6.b Two concentric conducting spheres with radii a, b $(b > a)$

Find the electric field between the spheres:

$$
\begin{aligned}
\oint \vec{E} \cdot d\vec{A} &= \frac{Q}{\varepsilon_0} \\
E \cdot 4\pi r^2 &= \frac{Q}{\varepsilon_0}
\end{aligned}
$$

$$
\begin{aligned}
V = \int_a^b E dr &= \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2} dr \\
&= -\frac{Q}{4\pi\varepsilon_0} \frac{1}{r}\bigg|_a^b \\
&= \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{a} - \frac{1}{b}\right]
\end{aligned}
$$

$$
\begin{aligned}
C = \frac{Q}{V} &= \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]} \\
&= \boxed{4\pi\varepsilon_0 \frac{ab}{b-a} = C}
\end{aligned}
$$

1.6.c Two concentric conducting cylinders of length L, large compared to their radii a, b $(b > a)$

Find the electric field between the cylinders:

$$
\begin{aligned}
\oint \vec{E} \cdot d\vec{A} &= \frac{Q}{\varepsilon_0} \\
E \cdot 2\pi r L &= \frac{Q}{\varepsilon_0} \\
E &= \frac{Q}{2\pi\varepsilon_0 L r}
\end{aligned}
$$

$$
\begin{aligned}
V = \int_a^b E dr &= \int_a^b \frac{Q}{2\pi\varepsilon_0 L} \frac{1}{r} \\
&= \frac{Q}{2\pi\varepsilon_0 L} \ln r\bigg|_a^b \\
&= \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right)
\end{aligned}
$$

$$
\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}}
$$

Problem 1.9

Calculate the attractive force between conductors in the parallel place capacitor (problem 1.6.a) and the parallel cylinder capacitor (problem 1.7) for:

1.9.a Fixed charges on each conductor

The electric field due to a parallel plate is:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$
$$E \cdot 2A = \frac{Q}{\varepsilon_0}$$
$$E = \frac{Q}{2\varepsilon_0 A}$$

$$F = QE = \boxed{\frac{Q^2}{2\varepsilon_0 A} = F}$$

The electric field due to a cylinder is:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$
$$E \cdot 2\pi r L = \frac{Q}{\varepsilon_0}$$
$$E = \frac{Q}{2\pi\varepsilon_0 Lr}$$

$$F = QE = \frac{Q^2}{2\pi\varepsilon_0 Lr}$$

Since the cylinders are a distance d apart, the force on one cylinder due to the other is:

$$\boxed{F = \frac{Q^2}{2\pi\varepsilon_0 Ld}}$$

1.9.b Fixed potential difference between the conductors

Recall that the voltage between the parallel plates is:

$$V = \frac{Qd}{\varepsilon_0 A}$$
$$\therefore Q = \frac{\varepsilon_0 AV}{d}$$

Substitituting Q into the force we found in part a:

$$F = \frac{\left(\frac{\varepsilon_0 AV}{d}\right)^2}{2\varepsilon_0 A}$$

$$\boxed{F = \frac{\varepsilon_0 A V^2}{d^2}}$$

From problem 1.7:

$$C \approx \frac{\pi\varepsilon_0}{\ln(d/a)}$$

$$\begin{aligned} V &= \frac{Q}{C} \\ &= \frac{Q}{\pi\varepsilon_0} \ln\left(\frac{d}{a}\right) \\ \therefore Q &= \frac{\pi\varepsilon_0 V}{\ln(d/a)} \end{aligned}$$

Substitituting Q into the force we found in part a:

$$F = \frac{\left(\frac{\pi\varepsilon_0 V}{\ln(d/a)}\right)^2}{2\pi\varepsilon_0 L d}$$

$$\boxed{F = \frac{\pi\varepsilon_0 V^2}{2Ld[\ln(d/a)]^2}}$$