Physics 505 **Fall 2007**

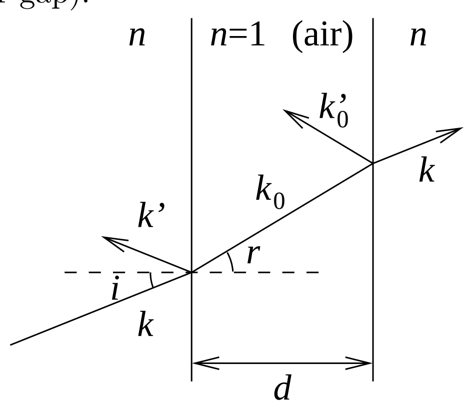
Homework Assignment #11 — Solutions

Textbook problems: Ch. 7: 7.3, 7.4, 7.6, 7.8

7.3 Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is indicent on the gap from one of the slabs with angle of indicence i. For linear polarization *both* parallel to *and* perpendicular to the plane of incidence,

a) calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power;

We introduce (complex) electric field vectors of the form $\vec{E}_i e^{i\vec{k}\cdot\vec{x}}$ and $\vec{E}_r e^{i\vec{k}'\cdot\vec{x}}$ on the incident side, $\vec{E}_+ e^{i\vec{k}_0\cdot\vec{x}}$ and $\vec{E}_- e^{i\vec{k}_0'\cdot\vec{x}}$ in the air gap, and $\vec{E}_t e^{i\vec{k}\cdot(\vec{x}-\vec{d})}$ on the transmitted side. (We have removed an unimportant phase from the transmitted side by shifting $\vec{x}$ by the vector $\vec{d}$ pointing from the incident to the transmitted side of the air gap).



If i is the incident angle, then the angle r from the normal in the air gap is given by Snell's law, $n \sin i = \sin r$, and the transmitted angle is also i (because it is the same dielectric). We see that

$$\cos r = \sqrt{1 - \sin^2 r} = \sqrt{1 - n^2 \sin^2 i}$$

and that $\cos r$ is purely imaginary in the event that i is greater than the critical angle for total internal reflection. To obtain E_t and E_r in terms of E_i, we may match the parallel components of $\vec{E}$ as well as the parallel components of $\vec{H}$. We consider two cases.

For $\vec{E}$ perpendicular to the plane of incidence, the matching becomes

$$\begin{array}{lll} & \text{first interface} & \text{second interface} \\ E^{\parallel}: & E_i + E_r = E_+ + E_-, & E_+ e^{i\phi} + E_- e^{-i\phi} = E_t \\ H^{\parallel}: & n(E_i - E_r)\cos i = (E_+ - E_-)\cos r, & (E_+ e^{i\phi} - E_- e^{-i\phi})\cos r = nE_t \cos i \end{array}$$

where we have introduced the phase

$$\phi = \vec{k}_0 \cdot \vec{d} = k_0 d \cos r = \frac{\omega d \cos r}{c}$$

The matching conditions at the first interface may be written as

$$\begin{aligned} E_+ &= \tfrac{1}{2} E_i (1+\alpha) + \tfrac{1}{2} E_r (1-\alpha) \\ E_- &= \tfrac{1}{2} E_i (1-\alpha) + \tfrac{1}{2} E_r (1+\alpha) \end{aligned} \tag{1}$$

where we have defined

$$\alpha = \frac{n \cos i}{\cos r} = \frac{n \cos i}{\sqrt{1 - n^2 \sin^2 i}}$$

Similarly, the matching conditions at the second interface yield

$$\begin{aligned} E_+ &= \tfrac{1}{2} e^{-i\phi} E_t (1+\alpha) \\ E_- &= \tfrac{1}{2} e^{i\phi} E_t (1-\alpha) \end{aligned} \tag{2}$$

Equating (1) and (2) allows us to solve for the ratios

$$\begin{aligned} \frac{E_t}{E_i} &= \frac{4\alpha}{(1+\alpha)^2 e^{-i\phi} - (1-\alpha)^2 e^{i\phi}} = \frac{2\alpha}{2\alpha \cos\phi - i(1+\alpha^2)\sin\phi} \\ \frac{E_r}{E_i} &= \frac{(1-\alpha^2)(e^{i\phi} - e^{-i\phi})}{(1+\alpha)^2 e^{-i\phi} - (1-\alpha)^2 e^{i\phi}} = \frac{i(1-\alpha^2)\sin\phi}{2\alpha \cos\phi - i(1+\alpha^2)\sin\phi} \end{aligned} \tag{3}$$

where

$$\alpha = \frac{n \cos i}{\sqrt{1 - n^2 \sin^2 i}}, \qquad \phi = \frac{\omega d \cos r}{c} = \frac{\omega d \sqrt{1 - n^2 \sin^2 i}}{c}$$

So long as i is below the critical angle, both α and ϕ are real. In this case, the transmission and reflection coefficients are

$$\begin{aligned} T &= \left| \frac{E_t}{E_i} \right|^2 = \frac{4\alpha^2}{4\alpha^2 \cos^2\phi + (1+\alpha^2)^2 \sin^2\phi} = \frac{4\alpha^2}{4\alpha^2 + (1-\alpha^2)^2 \sin^2\phi} \\ R &= \left| \frac{E_r}{E_i} \right|^2 = \frac{(1-\alpha^2)^2 \sin^2\phi}{4\alpha^2 \cos^2\phi + (1+\alpha^2)^2 \sin^2\phi} = \frac{(1-\alpha^2)^2 \sin^2\phi}{4\alpha^2 + (1-\alpha^2)^2 \sin^2\phi} \end{aligned} \tag{4}$$

Note that $T + R = 1$, as expected. However, this exhibits a classic interference behavior, where T oscillates between $(2\alpha/(1+\alpha^2))^2$ and 1 as the number of wavelengths in the gap vary.

For $\vec{E}$ parallel to the plane of incidence, we find instead the matching conditions

$$\begin{array}{lll} & \text{first interface} & \text{second interface} \\ E^{\parallel}: & (E_i - E_r)\cos i = (E_+ - E_-)\cos r, & (E_+ e^{i\phi} - E_- e^{-i\phi})\cos r = E_t \cos i \\ H^{\parallel}: & n(E_i + E_r) = (E_+ + E_-), & E_+ e^{i\phi} - E_- e^{-i\phi} = n E_t \end{array}$$

These equations have the same structure as the perpendicular case, but with the index of refraction entering somewhat differently. We find the matching conditions

$$\begin{aligned} n^{-1}E_+ &= \tfrac{1}{2}E_i(1+\beta) + \tfrac{1}{2}E_r(1-\beta) \\ n^{-1}E_- &= \tfrac{1}{2}E_i(1-\beta) + \tfrac{1}{2}E_r(1+\beta) \end{aligned}$$

and

$$\begin{aligned} n^{-1}E_+ &= \tfrac{1}{2}e^{-i\phi}E_t(1+\beta) \\ n^{-1}E_- &= \tfrac{1}{2}e^{i\phi}E_t(1-\beta) \end{aligned}$$

where this time

$$\beta = \frac{\cos i}{n\cos r} = \frac{\cos i}{n\sqrt{1-n^2\sin^2 i}}$$

These expressions are similar to (1) and (2) above, except with the replacement $E_\pm \to n^{-1}E_\pm$ and $\alpha \to \beta$. Hence the transmission and reflection coefficients are given by expressions identical to (4), except with the replacement $\alpha \to \beta$.

b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

To be concrete, consider the case for $\vec{E}$ perpendicular to the plane of incidence. Since i is greater than the critical angle, both α and ϕ will be purely imaginary. Whatever values they are, define

$$\alpha = i\gamma, \qquad \phi = i\xi$$

Then the ratios E_t/E_i and E_r/E_i in (3) become

$$\begin{aligned} \frac{E_t}{E_i} &= \frac{2i\gamma}{2i\gamma\cosh\xi + (1-\gamma^2)\sinh\xi} \\ \frac{E_r}{E_i} &= \frac{-(1+\gamma^2)\sinh\xi}{2i\gamma\cosh\xi + (1-\gamma^2)\sinh\xi} \end{aligned}$$

so that

$$\begin{aligned} T &= \left|\frac{E_t}{E_i}\right|^2 = \frac{4\gamma^2}{4\gamma^2 + (1+\gamma^2)^2\sinh^2\xi} \\ R &= \left|\frac{E_r}{E_i}\right|^2 = \frac{(1+\gamma^2)^2\sinh^2\xi}{4\gamma^2 + (1+\gamma^2)^2\sinh^2\xi} \end{aligned}$$

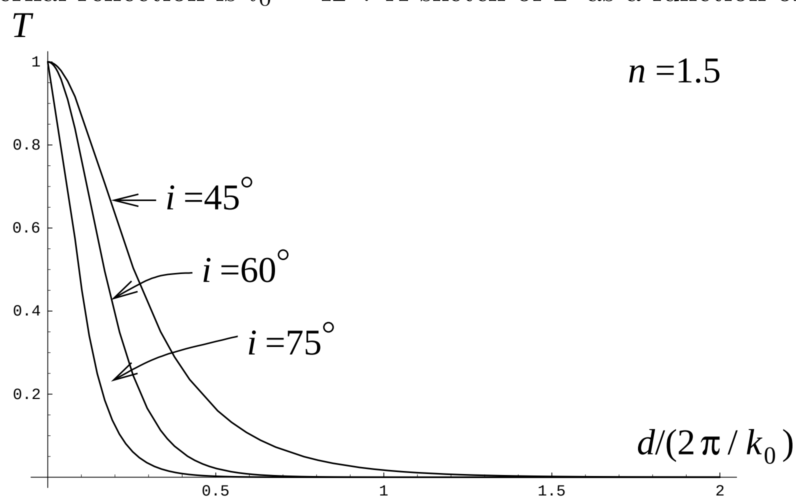
where

$$\gamma = -\frac{n\cos i}{\sqrt{n^2\sin^2 i - 1}}, \qquad \xi = \frac{\omega d\sqrt{n^2\sin^2 i - 1}}{c}$$

In this case, there is no oscillatory behavior in the transmitted power, but only exponential suppression as the air gap is widened. It is easy to see that $T \to 1$

when $d \to 0$ (corresponding to $\xi \to 0$) and that T falls exponentially to 0 when $d \to \infty$ (which is the same as $\xi \to \infty$).

For $n = 1.5$ (approximately the index of refraction of glass), the critical angle for total internal reflection is $i_0 \approx 42°$. A sketch of T as a function of d looks like



7.4 A plane-polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a nonpermeable medium of conductivity σ and dielectric constant ϵ.

a) Calculate the amplitude and phase of the reflected wave relative to the incident wave for arbitrary σ and ϵ.

A medium of dielectric constant ϵ and conductivity σ may be described by an effective dielectric constant

$$\varepsilon = \epsilon + i\frac{\sigma}{\omega}$$

Since the medium is nonpermeable, we have $\mu = \mu_0$. As a result, for normal incidence, the ratio of the reflected to the incident electric field is

$$\frac{E_r}{E_i} = \frac{1-n}{1+n}$$

where

$$n = \sqrt{\frac{\varepsilon}{\epsilon_0}} = \sqrt{\frac{\epsilon}{\epsilon_0} + i\frac{\sigma}{\epsilon_0\omega}}$$

The amplitude A and phase φ of the reflected wave is defined by

$$Ae^{i\varphi} = \frac{E_r}{E_i} = \frac{1 - \sqrt{\epsilon/\epsilon_0 + i\sigma/\epsilon_0\omega}}{1 + \sqrt{\epsilon/\epsilon_0 + i\sigma/\epsilon_0\omega}}$$

This expression implicitly defines A and φ. To be more explicit, we decompose the complex index of refraction into a magnitude and phase

$$n = \sqrt{\eta}e^{i\alpha/2}$$

where

$$\eta = \sqrt{\left(\frac{\epsilon}{\epsilon_0}\right)^2 + \left(\frac{\sigma}{\epsilon_0\omega}\right)^2}, \qquad \tan\alpha = \frac{\sigma}{\epsilon\omega} \tag{5}$$

The amplitude of the reflected wave is then

$$A = \sqrt{\frac{(1-n)(1-n^*)}{(1+n)(1+n^*)}} = \sqrt{\frac{1+|n|^2-2\Re n}{1+|n|^2+2\Re n}} = \sqrt{\frac{1+\eta-2\sqrt{\eta}\cos(\alpha/2)}{1+\eta+2\sqrt{\eta}\cos(\alpha/2)}} \tag{6}$$

while the phase is

$$\varphi = \arg\frac{1-n}{1+n} = \arg\frac{(1-n)(1+n^*)}{(1+n)(1+n^*)} = \tan^{-1}\frac{-2\Im n}{1-|n|^2} = \tan^{-1}\frac{-2\sqrt{\eta}\sin(\alpha/2)}{1-\eta} \tag{7}$$

Note that some care must be taken when extracting the phase. In particular, for $\varphi = \tan^{-1}(y/x)$, we must ensure that the angle φ lies in the proper quadrant defined by the point (x, y). This is why we choose to keep the minus sign in the numerator inside the arctan. Since the numerator is always negative, φ must lie in either the 3rd or the 4th quadrant. For $\eta > 1$, which is the case for all conventional dielectrics, φ lies in the 3rd quadrant. To highlight this, we may write

$$\varphi = \pi + \tan^{-1}\frac{2\sqrt{\eta}\sin(\alpha/2)}{\eta-1} \tag{8}$$

b) Discuss the limiting cases of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately

$$R \approx 1 - 2\frac{\omega}{c}\delta$$

where δ is the skin depth.

We begin with the case of a very poor conductor, $\sigma \ll \epsilon\omega$. In this case, η and α in (5) simplify to

$$\eta \approx \frac{\epsilon}{\epsilon_0}, \qquad \alpha \approx \frac{\sigma}{\epsilon\omega}$$

where we have kept only linear terms in σ. Substituting this into (6) and (8) gives

$$A \approx \left|\frac{1-\sqrt{\eta}}{1+\sqrt{\eta}}\right| = \left|\frac{1-\bar{n}}{1+\bar{n}}\right|$$

$$\varphi \approx \pi + \frac{\sqrt{\eta}}{\eta-1}\frac{\sigma}{\epsilon\omega} = \pi + \frac{\bar{n}}{\bar{n}^2-1}\frac{\sigma}{\epsilon\omega}$$

where $\bar{n} = \sqrt{\epsilon/\epsilon_0}$. Here we have assumed that $\bar{n} > 1$ and that $\sigma/\epsilon\omega \ll \bar{n} - 1$.

For a very good conductor, we take the opposite limit, $\sigma \gg \epsilon\omega$. In this case

$$\eta \approx \frac{\sigma}{\epsilon_0\omega} \gg 1, \qquad \alpha \approx \frac{\pi}{2}$$

Inserting this into (6) and (8) gives

$$A \approx \sqrt{\frac{\eta - \sqrt{2\eta} + 1}{\eta + \sqrt{2\eta} + 1}} \approx \left(\frac{1 - \sqrt{2/\eta}}{1 + \sqrt{2/\eta}} \right)^{1/2} \approx 1 - \sqrt{\frac{2}{\eta}} = 1 - \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}$$

$$\varphi \approx \pi + \tan^{-1} \frac{\sqrt{2\eta}}{\eta - 1} \approx \pi + \tan^{-1} \sqrt{\frac{2}{\eta}} \approx \pi$$

The amplitude A may be rewritten in terms of the skin depth $\delta = \sqrt{2/\mu_0 \sigma \omega}$

$$A \approx 1 - \omega\delta\sqrt{\mu_0\epsilon_0} = 1 - \frac{\omega}{c}\delta$$

This gives the reflection coefficient

$$R = A^2 \approx 1 - 2\frac{\omega}{c}\delta \tag{9}$$

7.6 A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction $n(\omega)$ $[n^2(\omega) = \epsilon(\omega)/\epsilon_0]$.

a) Show that the ratio of reflected power to incident power is

$$R = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2$$

while the ratio of power transmitted into the medium to the incident power is

$$T = \frac{4\Re n(\omega)}{|1 + n(\omega)|^2}$$

While this problem involves a complex dielectric constant $\epsilon(\omega)$, we note that the matching conditions for incident and reflected waves at an interface hold for arbitrary (including complex) values of μ and ϵ. For normal incidence, the expressions are simply

$$\frac{E_r}{E_i} = \frac{1 - n(\omega)}{1 + n(\omega)}, \qquad \frac{E_t}{E_i} = \frac{2}{1 + n(\omega)}$$

where we have furthermore assumed that the material is non-permeable so that $\mu = \mu_0$. For harmonic waves, the power is obtained from the real part of the Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon^*}{\mu^*}}|\vec{E}|^2\hat{n} \tag{10}$$

The reflection coefficient is then straightforward

$$R = \frac{\Re(\hat{n}\cdot\vec{S}_r)}{\Re(\hat{n}\cdot\vec{S}_i)} = \left|\frac{E_r}{E_i}\right|^2 = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$$

For the transmission coefficient, we also have to account for the different material

$$T = \frac{\Re(\hat{n}\cdot\vec{S}_t)}{\Re(\hat{n}\cdot\vec{S}_i)} = \Re\sqrt{\frac{\epsilon(\omega)^*}{\epsilon_0}}\left|\frac{E_t}{E_i}\right|^2 = \frac{4\Re[n(\omega)]}{|1+n(\omega)|^2}$$

b) Evaluate $\Re[i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)/2]$ as a function of (x,y,z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T.

We write the electric and magnetic fields inside the material as

$$\vec{E} = \vec{E}_t e^{ik\hat{n}\cdot\vec{x}}, \qquad \vec{B} = \sqrt{\mu_0\epsilon_0}\,n(\omega)\hat{n}\times\vec{E}_t e^{ik\hat{n}\cdot\vec{x}}$$

where the complex wavenumber k is given by

$$k(\omega) = \frac{\omega n(\omega)}{c}$$

In this case, the power per unit volume expression becomes

$$\begin{aligned}
\Re\left[\frac{i\omega}{2}(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)\right] &= \Re\left[\frac{i\omega}{2}\left(\epsilon(\omega)^*|\vec{E}|^2 - \frac{1}{\mu_0}|\vec{B}|^2\right)\right] \\
&= \Re\left[\frac{i\omega}{2}(\epsilon(\omega)^* - \epsilon_0|n(\omega)|^2)|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}}\right] \\
&= \Re\left[\frac{i\epsilon_0\omega}{2}(n(\omega)^{2\,*} - |n(\omega)|^2)|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}}\right] \\
&= \frac{\epsilon_0\omega\Im[n(\omega)^2]}{2}|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}} \\
&= \epsilon_0\omega\Re[n(\omega)]\Im[n(\omega)]|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}} \\
&= \sqrt{\frac{\epsilon_0}{\mu_0}}\Re[n(\omega)]\Im[k(\omega)]|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}}
\end{aligned} \tag{11}$$

The power per area transmitted into the material may then be calculated as

$$\begin{aligned}
P_t/A &= \int_0^\infty \Re\left[\frac{i\omega}{2}(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)\right]dz \\
&= \sqrt{\frac{\epsilon_0}{\mu_0}}\Re[n(\omega)]\Im[k(\omega)]|\vec{E}_t|^2\int_0^\infty e^{-2\Im[k(\omega)]z}dz \\
&= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\Re[n(\omega)]|\vec{E}_t|^2
\end{aligned}$$

On the other hand, the incident power per area may be obtained from (10)

$$P_i/A = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{E}_i|^2 \tag{12}$$

This gives

$$\frac{P_t}{P_i} = \Re[n(\omega)]\left|\frac{E_t}{E_i}\right|^2 = \Re[n(\omega)] \times \frac{4}{|1+n(\omega)|^2}$$

which agrees with the above calculation of the transmission coefficient.

Note that the complex Poynting vector inside the material is

$$\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}n(\omega)^*|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}}\hat{n}$$

Hence

$$\Re(\vec{\nabla}\cdot\vec{S}\,) = -\sqrt{\frac{\epsilon_0}{\mu_0}}\Re[n(\omega)]\Im[k(\omega)]|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}} \tag{13}$$

Comparing this with (11) demonstrates that the real part of the complex Poynting's theorem holds

$$\vec{\nabla}\cdot\vec{S} + \frac{i\omega}{2}(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*) + \frac{1}{2}\vec{J}^*\cdot\vec{E} = 0$$

so long as we take $\vec{J} = 0$ (ie no free currents).

c) For a conductor, with $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, σ real, write out the results of parts a and b in the limit $\epsilon_0\omega \ll \sigma$. Express your answer in terms of δ as much as possible. Calculate $\frac{1}{2}\Re(\vec{J}^*\cdot\vec{E})$ and compare with the result of part b. Do both enter the complex form of Poynting's theorem?

For a conductor with $\sigma \gg \omega\epsilon_0$, we make the approximation

$$n = \sqrt{1 + i\frac{\sigma}{\omega\epsilon_0}} \approx (1+i)\sqrt{\frac{\sigma}{2\omega\epsilon_0}} = (1+i)\frac{c}{\omega\delta}$$

where we have introduced the skin depth $\delta = \sqrt{2/\mu_0\sigma\omega}$. In this case, the reflection coefficient is approximately

$$\begin{aligned} R &= \left|\frac{1-n}{1+n}\right|^2 = \left|\frac{1-n^{-1}}{1+n^{-1}}\right|^2 \approx \left|\frac{1-(1-i)\omega\delta/2c}{1+(1-i)\omega\delta/2c}\right|^2 \approx \left|1-(1-i)\frac{\omega\delta}{c}\right|^2 \\ &\approx \left|1-2(1-i)\frac{\omega\delta}{c}\right| = \left|\left(1-2\frac{\omega\delta}{c}\right)+2i\frac{\omega\delta}{c}\right| \approx 1-2\frac{\omega\delta}{c} \end{aligned}$$

Not surprisingly, this is the same result as (9). The transmission coefficient is approximately

$$T = \frac{4\Re\, n}{|1+n|^2} \approx \frac{4c/\omega\delta}{|1+(1+i)c/\omega\delta|^2} \approx \frac{4c/\omega\delta}{|(1+i)c/\omega\delta|^2} = \frac{2\omega\delta}{c} \tag{14}$$

Note that $R + T \approx 1$ as expected.

For the power per unit volume of part b, we have from (11)

$$\begin{aligned}\Re\left[\frac{i\omega}{2}(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)\right] &= \epsilon_0\omega\Re(n)\Im(n)|\vec{E}_t|^2 e^{-2\Im(k)\hat{n}\cdot\hat{x}} \\ &\approx \epsilon_0\omega\left(\frac{c}{\omega\delta}\right)^2 |\vec{E}_t|^2 e^{-2\hat{n}\cdot x/\delta} \\ &= \frac{\epsilon_0 c^2}{\omega\delta^2}|\vec{E}_t|^2 e^{-2\hat{n}\cdot\vec{x}/\delta}\end{aligned}$$

Integrating this along z gives a power per area transmitted into the conductor

$$P_t/A = \int_0^\infty \frac{\epsilon_0 c^2}{\omega\delta^2}|\vec{E}_t|^2 e^{-2z/\delta}dz = \frac{\epsilon_0 c^2}{2\omega\delta}|\vec{E}_t|^2$$

Comparing this with the incident power per area (12) gives

$$\frac{P_t}{P_i} = \frac{c}{\omega\delta}\left|\frac{E_t}{E_i}\right|^2 = \frac{c}{\omega\delta}\frac{4}{|1+n|^2} \approx \frac{4c}{\omega\delta}\frac{1}{|n|^2} \approx \frac{4c}{\omega\delta}\frac{1}{|(1+i)c/\omega\delta|^2} = \frac{2\omega\delta}{c}$$

which agrees with the transmission coefficient (14).

For the divergence of the Poynting vector, note that (13) becomes

$$\Re(\vec{\nabla}\cdot\vec{S}) \approx -\frac{\epsilon_0 c^2}{\omega\delta^2}|\vec{E}_t|^2 e^{-2\hat{n}\cdot\vec{x}/\delta}$$

On the other hand, using $\vec{J} = \sigma\vec{E}$, we see that

$$\Re[\tfrac{1}{2}(\vec{J}^*\cdot\vec{E})] = \Re[\tfrac{1}{2}\sigma|\vec{E}|^2] = \tfrac{1}{2}\sigma|\vec{E}_t|^2 e^{-2\Im[k(\omega)]\hat{n}\cdot\vec{x}} = \frac{\epsilon_0 c^2}{\omega\delta^2}|\vec{E}_t|^2 e^{-2\hat{n}\cdot\vec{x}/\delta}$$

This expression gives the same value as the power per unit volume term $\Re[i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)/2]$. Hence if we include both the power per unit volume term and the work term $\Re[\frac{1}{2}(\vec{J}^*\cdot\vec{E})]$ in the complex Poynting's theorem

$$\vec{\nabla}\cdot\vec{S} + \frac{i\omega}{2}(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*) + \frac{1}{2}\vec{J}^*\cdot\vec{E} = 0$$

we would double count the contribution of the current, and this theorem would appear to be violated. To get the correct result, we recall that we have a choice of where the current $\vec{J}$ should be counted. In particular, by writing $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, we have played the trick of hiding the current $\vec{J}$ in the electric displacement

$$\vec{D} \quad\rightarrow\quad \vec{D}_{\text{eff}} = \vec{D} + \frac{i}{\omega}\vec{J} = \left(\epsilon_0 + \frac{i\sigma}{\omega}\right)\vec{E} = \epsilon_{\text{eff}}\vec{E}$$

In this case, the Ampère-Maxwell law becomes simply

$$\vec{\nabla} \times \vec{H} = -i\omega \vec{D}_{\text{eff}}$$

In particular, once we assume $\vec{D}_{\text{eff}} = \epsilon_{\text{eff}} \vec{E}$, we have set the explicit current to zero in this equation (although it is hidden in $\vec{D}_{\text{eff}}$). In this case, the 'correct' complex Poynting's theorem reads

$$\vec{\nabla} \cdot \vec{S} + \frac{i\omega}{2}(\vec{E} \cdot \vec{D}^*_{\text{eff}} - \vec{B} \cdot \vec{H}^*) = 0$$

where the work term is hidden in the power per unit volume term. Since expression (11) was actually calculated with $\vec{D}_{\text{eff}}$, this is the form of the Poynting's theorem that we have directly shown for the conductor.

On the other hand, if we treat the current $\vec{J}$ as an explicit quantity, then it enters through the work term $\Re[\frac{1}{2}(\vec{J}^* \cdot \vec{E})]$, but does not enter the power per unit volume term. In this case, we return to the full Poynting's theorem

$$\vec{\nabla} \cdot \vec{S} + \frac{i\omega}{2}(\vec{E} \cdot \vec{D}^* - \vec{B} \cdot \vec{H}^*) + \frac{1}{2}\vec{J}^* \cdot \vec{E} = 0$$

where $\vec{D}$ is the 'honest' electric displacement, without the addition of an effective current contribution. (For $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, we have $\vec{D} = \epsilon_0 \vec{E}$). Using this expression for $\vec{D}$ modifies the calculation of (11). In particular

$$\Re\left[\frac{i\omega}{2}(\vec{E} \cdot \vec{D}^* - \vec{B} \cdot \vec{H}^*)\right] = \Re\left[\frac{i\omega}{2}\left(\epsilon_0 |\vec{E}|^2 - \frac{1}{\mu_0}|\vec{B}|^2\right)\right] = 0$$

since ϵ_0 and μ_0 are both real. As a result, when treating $\vec{J}$ explicitly, power conservation is a balance between $\vec{\nabla} \cdot \vec{S}$ and $\frac{1}{2}\vec{J}^* \cdot \vec{E}$ only.

7.8 A monochromatic plane wave of frequency ω is indicdent normally on a stack of layers of various thicknesses t_j and lossless indices of refraction n_j. Inside the stack, the wave has both forward and backward moving components. The change in the wave through any interface and also from one side of a layer to the other can be described by means of 2×2 transfer matrices. If the electric field is written as

$$E = E_+ e^{ikx} + E_- e^{-ikx}$$

in each layer, the transfer matrix equation $E' = TE$ is explicitly

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}$$

a) Show that the transfer matrix for propagation inside, but across, a layer of index of refraction n_j and thickness t_j is

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} = I\cos(k_j t_j) + i\sigma_3 \sin(k_j t_j)$$

where $k_j = n_j\omega/c$, I is the unit matrix, and σ_k are the Pauli spin matrices of quantum mechanics. Show that the inverse matrix is T^*.

Normal incidence makes this problem straightforward. For a right moving plane wave of the form e^{ik_jz} passing through a layer of thickness t_j, one picks up a phase $e^{ik_jt_j}$, while for a left moving wave, one picks up a phase $e^{-ik_jt_j}$. More precisely

$$E'_+ = E_+(z = t_j) = E_+(z = 0)e^{ik_jt_j} = E_+e^{ik_jt_j}$$
$$E'_- = E_-(z = t_j) = E_-(z = 0)e^{-ik_jt_j} = E_-e^{-ik_jt_j}$$

This directly leads to the transfer matrix

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_jt_j} & 0 \\ 0 & e^{-ik_jt_j} \end{pmatrix}$$

where the inverse is obviously the complex conjugate.

b) Show that the transfer matrix to cross an interface from n_1 $(x < x_0)$ to n_2 $(x > x_0)$ is

$$T_{\text{interface}}(2, 1) = \frac{1}{2}\begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix} = I\frac{(n+1)}{2} - \sigma_1\frac{(n-1)}{2}$$

where $n = n_1/n_2$.

For the matching across layers, we take the $\vec{E}$ perpendicular to plane of incidence conventions. This gives simply

$$\begin{aligned} E^{\parallel}: &\qquad E_+ + E_- = E'_+ + E'_- \\ H^{\parallel}: &\qquad n_1(E_+ - E_-) = n_2(E'_+ - E'_-) \end{aligned}$$

which may be solved to give

$$E'_+ = \tfrac{1}{2}E_+(1+n) + \tfrac{1}{2}E_-(1-n)$$
$$E'_- = \tfrac{1}{2}E_+(1-n) + \tfrac{1}{2}E_-(1+n)$$

where $n = n_1/n_2$. This yields the transfer matrix

$$T_{\text{interface}}(2, 1) = \frac{1}{2}\begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix}$$

c) Show that for a complete stack, the incident, reflected, and transmitted waves are related by

$$E_{\text{trans}} = \frac{\det(T)}{t_{22}}E_{\text{inc}}, \qquad E_{\text{refl}} = -\frac{t_{21}}{t_{22}}E_{\text{inc}}$$

where t_{ij} are the elements of T, the product of the forward-going transfer matrices, including from the material filling space on the incident side into the first layer and from the last layer into the medium filling the space on the transmitted side.

It ought to be clear that the complete effect of going through several layers is to take a product of transfer matrices. For example

$$E' = TE, \quad \text{where} \quad T = \cdots T(4,3)T(n_3,t_3)T(3,2)T(n_2,t_2)T(2,1)$$

The transmitted and reflected electric fields are obtained by solving

$$\begin{pmatrix} E_t \\ 0 \end{pmatrix} = T \begin{pmatrix} E_i \\ E_r \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

This gives explicitly

$$E_t = t_{11}E_i + t_{12}E_r, \qquad 0 = t_{21}E_i + t_{22}E_r$$

which may be solved to obtain

$$E_r = -\frac{t_{21}}{t_{22}}E_i, \qquad E_t = \frac{t_{11}t_{22} - t_{12}t_{21}}{t_{22}}E_i = \frac{\det(T)}{t_{22}}E_i$$