Physics 505 **Fall 2007**

Homework Assignment #10 — Solutions

Textbook problems: Ch. 6: 6.1, 6.4, 6.13, 6.18

6.1 In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0, \vec{x}\,' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

a) Starting with the retarded solution to the three-dimensional wave equation (6.47), show that the source $f(\vec{x}\,', t) = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function [$\Theta(\xi) = 0$ (1) if $\xi < (>)\ 0$.]

Using

$$\Psi(\vec{x}, t) = \int \frac{[f(\vec{x}\,', t')]_{\text{ret}}}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

we find

$$\begin{aligned}\Psi(\vec{x}, t) &= \int \frac{\delta(x')\delta(y')\delta(t - R/c)}{R}\, dx'dy'dz' \\ &= \int_{-\infty}^{\infty} \frac{\delta(t - R/c)}{R}\, dz'\end{aligned}$$

where

$$R = |\vec{x} - \vec{x}\,'| = \sqrt{\rho^2 + (z - z')^2} \qquad \text{when } x' = y' = 0$$

By shifting $z' \to z' + z$, we end up with the integral

$$\Psi(\rho, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{\rho^2 + z'^2}/c)}{\sqrt{\rho^2 + z'^2}}\, dz' \tag{1}$$

Using

$$\delta(f(\zeta)) = \sum_i \frac{1}{|f'(\zeta)|}\delta(\zeta - \zeta_i) \tag{2}$$

where the sum is over the zeros of $f(\zeta)$, we see that

$$\delta(t - \sqrt{\rho^2 + z'^2}/c) = \sum_i \frac{c\sqrt{\rho^2 + z'^2}}{|z'|}\delta(z' - z'_i)$$

The zeros z'_i are given by

$$\rho^2 + z'^2 = c^2t^2 \qquad \Rightarrow \qquad z' = \pm\sqrt{c^2t^2 - \rho^2}$$

However it is clear that there are real zeros only if $c^2t^2 \geq \rho^2$ or $\rho < ct$. Going back to (1), and noting there are two zeros (one for each sign of the square root), we end up with

$$\Psi(\rho, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}$$

b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\Psi(x, t) = 2\pi c\Theta(ct - |x|)$$

For the sheet source, we use $f(\vec{x}\,', t') = \delta(x')\delta(t')$ to write

$$\Psi(\vec{x}, t) = \int \frac{\delta(x')\delta(t - R/c)}{R}\, dx'dy'dz'$$

where $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$. By integrating x' and shifting $y' \to y' + y$ and $z' \to z' + z$ we end up with

$$\Psi(x, t) = \int \frac{\delta(t - \sqrt{x^2 + y'^2 + z'^2}/c)}{\sqrt{x^2 + y'^2 + z'^2}}\, dy'dz' = \int \frac{\delta(t - \sqrt{\rho'^2 + x^2}/c)}{\sqrt{\rho'^2 + x^2}}\, \rho'd\rho'd\phi'$$

where we have gone to polar coordinates in the y'-z' plane. The ϕ' integral is now trivial. Treating the delta function as in (2) results in

$$\Psi(x, t) = 2\pi \int_0^\infty \sum_i c\delta(\rho' - \rho'_i)d\rho'$$

where the zeros ρ'_i corespond to

$$\rho'^2 + x^2 = c^2t^2 \qquad \Rightarrow \qquad \rho' = \pm\sqrt{c^2t^2 - x^2}$$

Since ρ' is non-negative, only the positive zero contributes, and we end up with

$$\Psi(x, t) = 2\pi c\Theta(ct - |z|)$$

where the step function enforces the condition for a real zero to exist.

6.4 A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi M R^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is nonrelativistic; the sphere has no excess charge on it.

a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2 R^3$.

We assume the sphere is magnetized and spinning along the $\hat{z}$ axis. Since the magnetic moment is $\vec{m} = \vec{M}V$ where $V = \frac{4}{3}\pi R^3$ is the volume of the sphere, we see that the magnetization is simply $\vec{M} = M\hat{z}$. As demonstrated earlier, a uniformly magnetized sphere has a uniform magnetic induction $\vec{B} = \frac{2}{3}\mu_0\vec{M}$ in its interior. In terms of m, this is

$$\vec{B} = \tfrac{2}{3}\mu_0\left(\frac{3}{4\pi R^3}m\hat{z}\right) = \frac{\mu_0 m}{2\pi R^3}\hat{z}$$

We now observe that the electric field $\vec{E}'$ in the rotating frame of the sphere may be related to lab quantities $\vec{E}$ and $\vec{B}$ by $\vec{E}' = \vec{E} + \vec{v}\times\vec{B}$. Ohm's law in the rotating reference frame is then $\vec{J} = \sigma\vec{E}' = \sigma(\vec{E} + \vec{v}\times\vec{B})$. Since no current flows in the steady state ($\vec{J} = 0$), this motion must induce an electric field $\vec{E} = -\vec{v}\times\vec{B}$. Using $\vec{\omega} = \omega\hat{z}$ and $\vec{v} = \vec{\omega}\times\vec{r}$, we obtain

$$\vec{E} = -(\vec{\omega}\times\vec{r})\times\vec{B} = -\frac{\mu_0 m\omega}{2\pi R^3}(\hat{z}\times\vec{r})\times\hat{z} = -\frac{\mu_0 m\omega}{2\pi R^3}(\vec{r} - \hat{z}(\hat{z}\cdot\vec{r}))$$

The vector structure is essentially a projection of $\vec{r}$ into the horizontal plane perpendicular to the $\hat{z}$ axis. In cylindrical coordinates, this indicates that

$$E_\rho = -\frac{\mu_0 m\omega\rho}{2\pi R^3} \qquad (3)$$

It is then a simple matter of applying Gauss' law to recover the volume charge density. However, before we do so, we note that this is a cylindrically symmetric electric field (pointed horizontally inward towards the $\hat{z}$ axis). It may at first be somewhat surprising that a sphere will give a cylindrical electric field. However, rotation about an axis is actually a cylindrical process. So from this point of view, the electric field is quite natural.

Using $\rho = \epsilon_0\vec{\nabla}\cdot\vec{E}$ we obtain a uniform volume charge density

$$\rho = \epsilon_0\frac{\partial E_\rho}{\partial\rho} = -\frac{\mu_0\epsilon_0 m\omega}{2\pi R^3} = -\frac{m\omega}{2\pi c^2 R^3}$$

It is important to note that, while the charge density is uniform inside the sphere, the electric field is *not* radial. The discrepancy between a uniform spherical charge distribution and the cylindrical electric field must then arise due to a surface charge. This then provides a hint as to how to approach the remainder of this problem.

b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has nonvanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{33}/2$.

No charge resides outside the sphere. As a result, the exterior field may be described through the multipole expansion. As indicated, charge neutrality guarantees the vanishing of the monopole ($l = 0$) moment. Furthermore, the odd moments vanish due to symmetry of the electric field (3) under the parity transformation $z \to -z$. (That is of course the internal field; however we may see that the external field must necessarily respect the symmetry of the internal one.) Thus a simple symmetry argument demonstrates that the lowest possible multipole is the quadrupole ($l = 2$). Symmetry alone will not preclude higher even moments. However an explicit calculation will.

Without knowing the surface charge, we cannot directly calculate the electric multipoles. However, we note that the interior electric field (3) can be integrated to obtain the interior electrostatic potential

$$\Phi(\rho) = -\int \vec{E} \cdot d\vec{\ell} = -\int E_\rho \, d\rho = \Phi_0 + \frac{\mu_0 m\omega\rho^2}{4\pi R^3}$$

Converted back to spherical coordinates, this gives

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{4\pi R^3} r^2 \sin^2\theta = \Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2 [P_0(\cos\theta) - P_2(\cos\theta)]$$

where we have converted $\sin^2\theta$ into Legendre polynomials. This can be written explicitly as a Legendre expansion

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3} r^2 P_2(\cos\theta)$$

so that in particular the potential at the surface of the sphere is

$$\Phi(R,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R}\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R} P_2(\cos\theta)$$

We may now solve for the exterior potential by treating this as an electrostatic boundary value problem. We recall that, given a sphere with azimuthally symmetric potential $V(\theta) = \sum_l \alpha_l P_l(\cos\theta)$ on the surface, the exterior solution has the form $\Phi(r,\theta) = \sum_l \alpha_l (R/r)^{l+1} P_l(\cos\theta)$. Furthermore, charge neutrality in the present case forces the monopole ($l = 0$) term to vanish. Hence we find that $\Phi_0 = -\mu_0 m\omega/6\pi R$, and that the external potential is

$$\Phi(r,\theta) = -\frac{\mu_0 m\omega R^2}{6\pi r^3} P_2(\cos\theta) \tag{4}$$

Incidentally, we could write an expression valid both in the interior and the exterior as

$$\Phi(r,\theta) = \frac{\mu_0 m\omega}{6\pi R}\left[\left(\frac{r^2}{R^2} - 1\right)\Theta(R-r)P_0(\cos\theta) - R\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right] \tag{5}$$

Note that this potential is only harmonic outside the sphere; inside the sphere the r^2/R^2 term multiplying $P_0(\cos\theta)$ is not of the right $(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$ form to be harmonic. However, this is present precisely because of the uniform volume charge density, which acts as a $l = 0$ source.

In any case, we are essentially done, as the exterior potential (4) clearly has only a quadrupole term

$$\Phi = -\sqrt{\frac{4\pi}{5}}\frac{\mu_0 m\omega R^2}{6\pi}\frac{Y_{2,0}(\theta,\phi)}{r^3}$$

Comparing this with the multipole expansion

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{l,m}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}}$$

gives

$$q_{2,0} = -4\pi\epsilon_0\sqrt{\frac{5}{4\pi}}\frac{\mu_0 m\omega R^2}{6\pi} = -\sqrt{\frac{5}{4\pi}}\frac{2m\omega R^2}{3c^2}$$

Converting to cartesian tensors yields

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}}q_{2,0} = -\frac{4m\omega R^2}{3c^2}, \qquad Q_{11} = Q_{22} = -\tfrac{1}{2}Q_{33}$$

c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-chrage density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

The surface charge may be computed by first obtaining the jump in the normal component of the electric field at the surface of the sphere. Working in spherical components, and taking the gradient of the potential (5), we find

$$\begin{aligned} E_r^{\text{out}} &= -\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta) \\ E_r^{\text{in}} &= -\frac{\mu_0 m\omega r}{3\pi R^3}[P_0(\cos\theta) - P_2(\cos\theta)] \end{aligned}$$

The surface charge is then computed as

$$\sigma = \epsilon_0(E_r^{\rm out} - E_r^{\rm in})\Big|_{r=R} = -\frac{\mu_0\epsilon_0 m\omega}{3\pi R^2}[\tfrac{3}{2}P_2(\cos\theta) - (P_0(\cos\theta) - P_2(\cos\theta))]$$
$$= \frac{m\omega}{3\pi c^2 R^2}[P_0(\cos\theta) - \tfrac{5}{2}P_2(\cos\theta)]$$

d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact (by any path) is $\mathcal{E} = \mu_0 m\omega/4\pi R$.

Although the sphere is rotating, both the electric and the magnetic field are static. Hence the line integral of the electric field gives simply the e lectrostatic potential. In this case

$$\mathcal{E} = \int_{\rm equator}^{\rm pole} \vec{E}\cdot d\vec{\ell} = \Phi_{\rm equator} - \Phi_{\rm pole} = \Phi(\theta = \tfrac{\pi}{2}) - \Phi(\theta = 0)$$

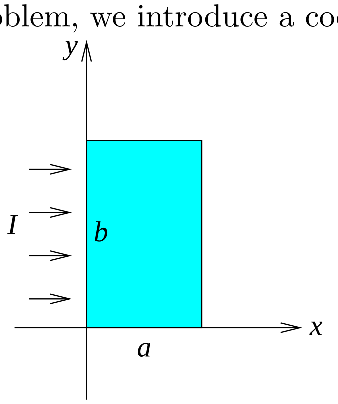
Using (4) or (5) evaluated on the surface, this becomes

$$\mathcal{E} = -\frac{\mu_0 m\omega}{6\pi R}[P_2(0) - P_2(1)] = \frac{\mu_0 m\omega}{4\pi R}$$

6.13 A parallel plate capacitor is formed of two flat rectangular perfectly conducting sheets of dimensions a and b separated by a distance d small compared to a or b. Current is fed in and taken out uniformly along the adjacent edges of length b. With the input current and voltage defined at this end of the capacitor, calculate the input impedance or admittance using the field concepts of Section 6.9.

a) Calculate the electric and magnetic fields in the capacitor correct to second order in powers of the frequency, but neglecting fringing fields.

To set up this problem, we introduce a coordinate system



where the two plates are separated by a distance d along the z axis (pointing out of the page). For an electrostatic system, the electric and magnetic fields in the capacitor are simple to write down

$$\vec{E} = -(V/d)\hat{z}, \qquad \vec{B} = 0$$

where we have assumed the top plate is positively charged, and where V denotes the voltage difference between the plates. These expressions neglect fringing, of course.

For a harmonically varying voltage with frequency ω, we may generalize the electric field expression to

$$\vec{E}^{(0)} = -(V/d)\cos(\omega t)\hat{z}$$

which we can write as the real part of

$$\vec{E}^{(0)} = -(V/d)e^{-i\omega t}\hat{z} \tag{6}$$

The $^{(0)}$ superscript denotes the 'simple' or zeroth order expression for the electric field. We now note that since this electric field is varying in time, it induces a displacement current, which in turn generates a magnetic field through the Ampère-Maxwell equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + (1/c^2)\partial\vec{E}/\partial t$. Since $\vec{J} = 0$ between the plates of the capacitor, this equation gives rise to

$$\vec{\nabla} \times \vec{B}^{(1)} = -\frac{i\omega}{c^2}\vec{E}^{(0)} = \frac{i\omega}{c^2}\frac{V}{d}\hat{z} \tag{7}$$

where we have converted to the frequency domain and where we have suppressed the harmonic factor $e^{-i\omega t}$. In order to determine the induced magnetic field $\vec{B}^{(1)}$, we now appeal to a symmetry argument. Since the current is uniformly fed in along the left side of the capacitor (ie independent of y), we expect the fields to be independent of y, so long as we do not get too close to the edges of the capacitor. We furthermore assume that the fields are independent of the z direction. This is certainly true for the zeroth order electric field (6). In addition, since the displacement current is independent of z, it is natural for $\vec{B}^{(1)}$ to also be independent of z. In this case, $\vec{B}^{(1)}$ can only depend on x, and in order to solve (7) it needs to point in the $\hat{y}$ direction

$$\vec{B}^{(1)} = \frac{i\omega}{c^2}\frac{V}{d}(x - x_0)\hat{y}$$

(Actually we can add to this a uniform magnetic field pointing in an arbitrary direction. However, any $\hat{x}$ or $\hat{z}$ component of the magnetic field would not obey the symmetry properties of this system, ignoring fringing.) Here x_0 is a constant of integration, and it cannot be eliminated by symmetry alone.

In order to figure out the constant x_0, we may related the magnetic field to the surface current density $\vec{K}$ for the current flowing in the plates of the capacitor. We assume the magnetic field is contained between the two plates of the capacitor. In this case, the current density on the top plate of the capacitor is given by

$$\vec{K} = \hat{n} \times \vec{H}$$

where $\hat{n}$ is the unit normal pointing from the top plate towards the bottom plate (ie $\hat{n} = -\hat{z}$). This gives

$$\vec{K} = -\hat{z} \times \frac{i\omega}{\mu_0 c^2} \frac{V}{d}(x - x_0)\hat{y} = \epsilon_0 i\omega \frac{V}{d}(x - x_0)\hat{x}$$

Physically this makes sense, since the surface current is flowing in the $\hat{x}$ direction in and out of the plates of the capacitor. We now use the fact that the current is fed in on the left side ($x = 0$). By current conservation, since charge cannot disappear off the right side of the capacitor ($x = a$), the current must go to zero when $x = a$. This requirement shows us that we must set the integration constant $x_0 = a$ in the above. As a result, we end up with

$$\vec{B}^{(1)} = \frac{i\omega}{c^2} \frac{V}{d}(x - a)\hat{y} \tag{8}$$

At this stage, it should not be a surprise to us to note that this oscillating magnetic field will in turn induce an electric field $\vec{E}^{(2)}$ through Faraday's law (for harmonic fields)

$$\vec{\nabla} \times \vec{E}^{(2)} = i\omega \vec{B}^{(1)} = -\frac{\omega^2}{c^2} \frac{V}{d}(x - a)\hat{y}$$

Assuming, by symmetry, that $\vec{E}^{(2)}$ points in the $\hat{z}$ direction, we find

$$\vec{E}^{(2)} = \frac{\omega^2}{c^2} \frac{V}{d}\hat{z} \int (x - a)dx = \frac{\omega^2}{c^2} \frac{V}{d}[\tfrac{1}{2}(x - a)^2 - C]\hat{z}$$

where C is a constant of integration. In order to fix this constant C, we note that we have defined the electric field $\vec{E}^{(0)}$ in (6) based on the voltage V measured at the left ($x = 0$) side of the capacitor. Since $\vec{E}^{(0)}(x = 0)$ is already the 'correct' value, we do not want $\vec{E}^{(2)}(x = 0)$ to add any correction to this. We thus demand $\vec{E}^{(2)}(x = 0) = 0$, and this fixes the constant to be $C = a^2/2$. We may then write

$$\vec{E}^{(2)} = \frac{\omega^2}{c^2} \frac{V}{2d}[(x - a)^2 - a^2]\hat{z} \tag{9}$$

To continue, we now note that the time varying $\vec{E}^{(2)}$ will create a magnetic field $\vec{B}^{(3)}$ through the Ampère-Maxwell equation

$$\vec{\nabla} \times \vec{B}^{(3)} = -\frac{i\omega}{c^2}\vec{E}^{(2)} = -\frac{i\omega^3}{c^4} \frac{V}{2d}[(x - a)^2 - a^2]\hat{z}$$

The solution to this is

$$\vec{B}^{(3)} = -\frac{i\omega^3}{c^4} \frac{V}{2d}[\tfrac{1}{3}(x - a)^3 - a^2(x - a)]\hat{y} \tag{10}$$

Obviously, this process can go on forever, and we end up with a series solution for the complete electric and magnetic fields. The reason this series converges is that we essentially have an expansion parameter of the form ω/c, which we consider to be small. (Actually, a more proper expansion parameter would be the dimensionless quantity $\omega a/c$.) Stopping at this order, we collect (6), (8), (9) and (10) to obtain

$$\begin{aligned}\vec{E} &= \vec{E}^{(0)} + \vec{E}^{(2)} + \cdots \approx -\frac{V}{d}\left(1 - \frac{\omega^2}{2c^2}[(x-a)^2 - a^2] + \cdots\right)\hat{z} \\ \vec{B} &= \vec{B}^{(1)} + \vec{B}^{(3)} + \cdots \approx \frac{iV}{d}\frac{\omega}{c^2}(x-a)\left(1 - \frac{\omega^2}{2c^2}[\tfrac{1}{3}(x-a)^2 - a^2] + \cdots\right)\hat{y}\end{aligned} \tag{11}$$

At this stage, we note that the Gauss' law equations $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$ are trivially satisfied as well. Restoring the $e^{-i\omega t}$ factor and taking the real part gives

$$\begin{aligned}\vec{E} &\approx -\frac{V}{d}\left(1 - \frac{\omega^2}{2c^2}[(x-a)^2 - a^2] + \cdots\right)\cos(\omega t)\hat{z} \\ \vec{B} &\approx \frac{V}{d}\frac{\omega}{c^2}(x-a)\left(1 - \frac{\omega^2}{2c^2}[\tfrac{1}{3}(x-a)^2 - a^2] + \cdots\right)\sin(\omega t)\hat{y}\end{aligned}$$

Note that this iterative procedure of using the Ampère-Maxwell law and Faraday's law is essentially equivalent to developing a perturbation series solution. We could also be more direct and note that the frequency domain expressions

$$\vec{\nabla}\times\vec{B} = -\frac{i\omega}{c^2}\vec{E}, \qquad \vec{\nabla}\times\vec{E} = i\omega\vec{B}$$

simplify under the assumptions

$$\vec{E} = E_z(x)\hat{z}, \qquad \vec{B} = B_y(x)\hat{y}$$

to become

$$B_y'(x) = -\frac{i\omega}{c^2}E_z(x), \qquad E_z'(x) = -i\omega B_y(x) \tag{12}$$

Taking an x derivative of the first equation and using the second then gives

$$B_y''(x) = -\frac{i\omega}{c}^2 E_z'(x) = -\frac{\omega^2}{c^2}B_y(x)$$

This is a standard 'harmonic oscillator' differential equation, and the solution may be written as

$$B_y = B_0 \sin\left[\frac{\omega}{c}(x-a)\right]$$

Note that we have chosen boundary conditions such that $B_y(x=a) = 0$. Substituting this magnetic field into the first equation of (12) then gives

$$E_z = icB_0\cos\left[\frac{\omega}{c}(x-a)\right]$$

Since B_0 is an integration constant, we may choose to rewrite these expressions as

$$E_z = -\frac{V}{d}\frac{\cos[\omega(x-a)/c]}{\cos(\omega a/c)}$$
$$B_y = i\frac{V}{cd}\frac{\sin[\omega(x-a)/c]}{\cos(\omega a/c)}$$

Note that $E_z(x=0) = -V/d$ and $B_y(x=a) = 0$. Expanding this for small ω reproduces the series solution of (11).

Finally, note that we may calculate the surface charge density on the top plate by

$$\sigma = -\epsilon_0 E_z = \frac{\epsilon_0 V}{d}\frac{\cos[\omega(x-a)/c]}{\cos(\omega a/c)}$$

and the surface current density by

$$\vec{K} = -\hat{z}\times\vec{H} = \frac{\epsilon_0 ciV}{d}\frac{\sin[\omega(x-a)/c]}{\cos(\omega a/c)}\hat{x}$$

We can verify that this satisfies the frequency domain current conservation law

$$\vec{\nabla}\cdot\vec{J} - i\omega\rho = 0$$

where $\vec{J} = \vec{K}\delta(z-d)$ and $\rho = \sigma\delta(z-d)$ for the top plate located at $z = d$.

Note that we have been a bit sloppy about specifying how the current is fed into the capacitor on the left side of the plates. Our implicit assumption is that the current is fed in the top plate and taken out of the bottom plate in such a way that the magnetic fields created by the input wires are negligible. As a result, we assume all electric and magnetic fields are essentially vanishing except in the volume between the plates. If we had fields outside of the plates, then there may be additional contributions to the surface charge density and surface current density expressions of the form

$$\sigma = -\hat{z}\cdot(\vec{E}_{\text{in}} - \vec{E}_{\text{out}}), \qquad \vec{K} = -\hat{z}\times(\vec{H}_{\text{in}} - \vec{H}_{\text{out}})$$

b) Show that the expansion of the reactance (6.140) in powers of the frequency to an appropriate order is the same as that obtained for a lumped circuit consisting of a capacitance $C = \epsilon_0 ab/d$ in series with an inductance $L = \mu_0 ad/3b$.

The expression for reactance is given by

$$X \approx \frac{4\omega}{|I_i|^2}\int_V (w_m - w_e)d^3x \tag{13}$$

To lowest nontrivial order, we use (6) and (8) to write

$$w_e = \frac{\epsilon_0}{4}|\vec{E}|^2 \approx \frac{\epsilon_0|V_i|^2}{4d^2}, \qquad w_m = \frac{1}{4\mu_0}|\vec{B}|^2 \approx \frac{\omega^2|V_i|^2}{4\mu_0 c^4 d^2}(x-a)^2 \tag{14}$$

These expressions are given in terms of the input voltage V_i. However we would like to rewrite them in terms of the current I_i. To do so, we note that given (6)

$$\vec{E} \approx -\frac{V_i}{d}\hat{z}$$

the surface charge density on the top plate is

$$\sigma = -\epsilon_0 \hat{z} \cdot \vec{E} \approx \frac{\epsilon_0 V_i}{d}$$

At this level of approximation, these expressions are identical to the electrostatic case. In particular, σ is approximately uniform across the top plate, so the total charge on the capacitor is given by

$$Q = \sigma \times (\text{Area}) \approx \frac{\epsilon_0 V_i ab}{d}$$

Although this looks identical to the static expression, we should keep in mind that all the quantities we are writing down here are actually harmonic (ie they should be multiplied by $e^{-i\omega t}$). As a result, we obtain the current

$$I_i = \frac{dQ}{dt} = -i\omega Q \approx -\frac{i\omega\epsilon_0 V_i ab}{d}$$

Solving this for V_i and substituting into (14) results in

$$w_e \approx \frac{|I_i|^2}{4\epsilon_0\omega^2 a^2 b^2}, \qquad w_m \approx \frac{\mu_0 |I_i|^2}{4a^2b^2}(x-a)^2$$

Integrating the energy densities over the volume between the plates of the capacitor gives

$$\int w_e\, d^3x \approx \frac{|I_i|^2 d}{4\epsilon_0\omega^2 ab}$$
$$\int w_m\, d^3x \approx \frac{\mu_0|I_i|^2 d}{4a^2 b}\int_0^a (x-a)^2 dx = \frac{\mu_0|I_i|^2 ad}{12b}$$

Inserting these expressions into (13) gives a reactance

$$X \approx \frac{\mu_0\omega ad}{3b} - \frac{d}{\epsilon_0\omega ab}$$

Identifying $X = \omega L$ for an inductance and $X = -1/\omega C$ for a capacitance, we see that the above expression is equivalent to a series combination of a capacitor and inductor with

$$C = \frac{\epsilon_0 ab}{d}, \qquad L = \frac{\mu_0 ad}{3b}$$

6.18 Consider the Dirac expression

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\int_L \frac{d\vec{l}\,' \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

for the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

a) Calculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\phi = \frac{g(1-\cos\theta)}{4\pi r\sin\theta} = \left(\frac{g}{4\pi r}\right)\tan\frac{\theta}{2}$$

Taking the Dirac string along the negative $\hat{z}$ axis, we write $\vec{x}\,' = z'\hat{z}$ and $d\vec{\ell}' = \hat{z}dz'$. Hence Dirac's expression is

$$\begin{aligned}\vec{A}(\vec{x}\,) &= \frac{g}{4\pi}\int_{-\infty}^{0} dz'\,\frac{\hat{z}\times(\vec{x} - z'\hat{z})}{|\vec{x} - z'\hat{z}|^3}\\ &= \frac{g}{4\pi}\int_{-\infty}^{0} dz'\,\frac{\hat{z}\times\vec{x}}{[\rho^2 + (z-z')^2]^{3/2}}\\ &= \frac{g}{4\pi}(\hat{z}\times\vec{x})\int_{-\infty}^{-z}\frac{du}{(\rho^2+u^2)^{3/2}}\end{aligned}$$

This integral is easily performed by trig substitution. The result is

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\frac{\hat{z}\times\vec{x}}{\rho^2}\left(1 - \frac{z}{r}\right)$$

where $\rho^2 = x^2 + y^2$ and $r^2 = x^2 + y^2 + z^2$. Noting that $\hat{z}\times\vec{x} = \rho\hat{\phi} = r\sin\theta\hat{\phi}$, and converting to spherical coordinates, we obtain

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\frac{r-z}{r^2\sin\theta}\hat{\phi} = \frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}$$

b) Verify that $\vec{B} = \vec{\nabla}\times\vec{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$.

Note that the vector potential blows up on the negative $\hat{z}$ axis. (The positive $\hat{z}$ axis is safe, as a Taylor or l'Hopital expansion near $\theta = 0$ will demonstrate.) Away from this point, we have

$$\begin{aligned}\vec{B} = \vec{\nabla}\times\vec{A} &= \hat{r}\frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\phi) - \hat{\theta}\frac{1}{r}\partial_r(rA_\phi)\\ &= \hat{r}\frac{1}{r\sin\theta}\partial_\theta\left(\frac{g}{4\pi}\frac{1-\cos\theta}{r}\right) = \hat{r}\left(\frac{g}{4\pi r^2}\right)\end{aligned}$$

which is the expected field of a magnetic monopole.

c) With the $\vec{B}$ determined in part b, evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown in the figure. Consider $\theta < \pi/2$ and $\theta > \pi/2$ separately, but always calculate the upward flux.

Assuming $\vec{B} = g\hat{r}/4\pi r^2$ everywhere, the flux through a circular loop of radius $R\sin\theta$ is

$$\begin{aligned}\Phi &= \int \vec{B}\cdot\hat{n}\,da = \int B_z\,da = \frac{g}{4\pi}\int \frac{z}{(\rho^2+z^2)^{3/2}}\,\rho d\rho\,d\phi \\ &= \frac{gz}{4}\int_0^{(R\sin\theta)^2} \frac{du}{(u+z^2)^{3/2}} = -\frac{gz}{2}\frac{1}{\sqrt{u+z^2}}\bigg|_0^{(R\sin\theta)^2} \\ &= \frac{gR\cos\theta}{2}\left(\frac{1}{R|\cos\theta|}-\frac{1}{R}\right) = \frac{g}{2}\big(\mathrm{sgn}(\cos\theta)-\cos\theta\big)\end{aligned}$$

where we have used $z = R\cos\theta$. For $\theta < \pi/2$ (the top hemisphere) we find $\Phi_{\rm top} = \frac{g}{2}(1-\cos\theta)$, while for $\theta > \pi/2$ we find $\Phi_{\rm bottom} = \frac{g}{2}(-1-\cos\theta)$. Note that the (upward) flux so calculated is discontinuous as we pass through the plane of the monopole.

d) From $\oint \vec{A}\cdot d\vec{l}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part c. Show that they are equal for $0 < \theta < \pi/2$, but have a *constant* difference for $\pi/2 < \theta < \pi$. Interpret this difference.

By Stokes' theorem, the line integral of the vector potential gives the magnetic flux. We find

$$\oint \vec{A}\cdot d\vec{\ell} = \int_0^{2\pi} A_\phi(R,\theta)\,R\sin\theta\,d\phi = \frac{g}{4\pi}\frac{1-\cos\theta}{R\sin\theta}(2\pi R\sin\theta) = \frac{g}{2}(1-\cos\theta)$$

Thus

$$\oint \vec{A}\cdot d\vec{\ell} = \Phi_{\rm top} = \Phi_{\rm bottom} + g$$

What has happened in this case is that the computation of part c did not take into account the flux of the Dirac string. For a positively charged monopole, the Dirac string carries upward magnetic flux. So the total flux of the monopole plus string is really $\Phi_{\rm bottom} + g$. This is fully accounted for by taking the line integral of the vector potential (which is after all the vector potential due to the Dirac string).

Of course, an 'honest' magnetic monopole will have a magnetic field $\vec{B} = g\hat{r}/4\pi r^2$ *everywhere* in space. In this case, the flux calculation of part *c*) is the 'correct' one. Every calculation involving the vector potential must then be treated with care, and in particular the location of the Dirac string may have to be moved by gauge transformation when working with $\vec{A}$ in the southern hemisphere. In the modern language of differential geometry (fiber bundles), we have to introduce separate coordinate patches for the northern and southern hemisphere, with an overlap region around the equator. We then define differentiable transition functions (essentially gauge transformations) connecting the different sections of the bundle in the overlap region. The Dirac string can then be avoided by working with the fiber bundle itself.