Physics 505 **Fall 2007**

Homework Assignment #9 — Solutions

Textbook problems: Ch. 5: 5.19, 5.21, 5.22, 5.27

5.19 A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis.

a) Determine the magnetic field $\vec{H}$ and magnetic induction $\vec{B}$ at all points on the axis of the cylinder, both inside and outside.

We use a magnetic scalar potential and the expression

$$\Phi_M = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}\cdot\vec{M}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\hat{n}'\cdot\vec{M}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,da'$$

Orienting the cylinder along the z axis, we take a uniform magnetization $\vec{M} = M_0\hat{z}$. In this case the volume integral drops out, and the surface integral only picks up contributions on the endcaps. Thus

$$\Phi_M = \frac{M_0}{4\pi}\left[\int_{\text{top}} \frac{1}{|\vec{x}-\vec{x}\,'|}\,da' - \int_{\text{bottom}} \frac{1}{|\vec{x}-\vec{x}\,'|}\,da'\right]$$

where 'top' and 'bottom' denote $z = \pm L/2$, and the integrals are restricted to $\rho < a$. On axis ($\rho = 0$) we have simply

$$\begin{aligned}
\Phi_M(z) &= \frac{M_0}{4\pi}\int\left(\frac{1}{\sqrt{\rho^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho^2+(z+L/2)^2}}\right)\rho\,d\rho\,d\phi \\
&= \frac{M_0}{4}\int_0^{a^2}\left(\frac{1}{\sqrt{\rho^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho^2+(z+L/2)^2}}\right)d\rho^2 \\
&= \frac{M_0}{2}\left[\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - |z-L/2| + |z+L/2|\right]
\end{aligned}$$

On axis, the field can only point in the z direction. It is given by

$$\begin{aligned}
H_z = -\frac{\partial}{\partial z}\Phi_M = -\frac{M_0}{2}\Bigg[&\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} \\
&- \operatorname{sgn}(z-L/2) + \operatorname{sgn}(z+L/2)\Bigg]
\end{aligned}$$

Note that the last two terms cancel when $|z| > L/2$, but add up to 2 inside the magnet. Thus we may write

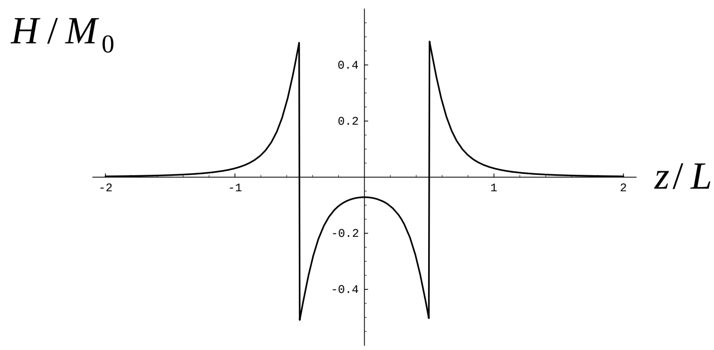
$$H_z = -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\,\Theta(L/2-|z|)\right]$$

where $\Theta(\xi)$ denotes the unit step function, $\Theta = 1$ for $\xi > 0$ (and 0 otherwise). The magnetic induction is obtained by rewriting the relation $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ as $\vec{B} = \mu_0(\vec{H} + \vec{M})$. Since the magnetization is only nonzero inside the magnet [ie $M_z = M_0\,\Theta(L/2 - |z|)$], the addition $\vec{H} + \vec{M}$ simply removes the step function term. We find

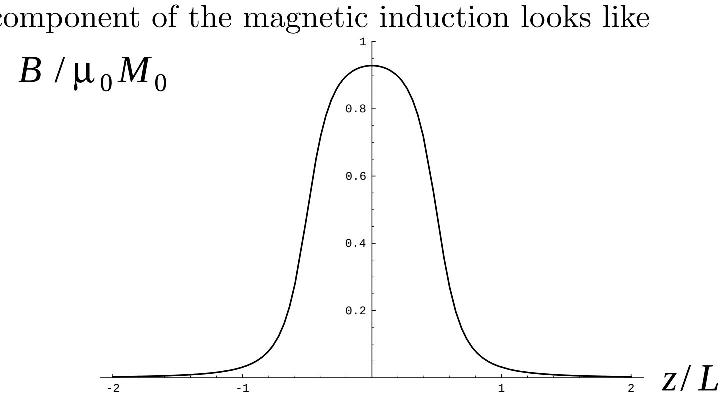
$$B_z = \mu_0(H_z + M_z) = -\frac{\mu_0 M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}}\right] \qquad (1)$$

b) Plot the ratios $\vec{B}/\mu_0 M_0$ and $\vec{H}/M_0$ on the axis as functions of z for $L/a = 5$.

The z component of the magnetic field looks like



while the z component of the magnetic induction looks like



Note that B_z is continuous, while H_z jumps at the ends of the magnet. This jump may be thought of as arising from effective magnetic surface charge.

5.21 A magnetostatic field is due entirely to a localized distribution of permanent magnetization.

a) Show that

$$\int \vec{B} \cdot \vec{H}\, d^3x = 0$$

provided the integral is taken over all space.

So long as the magnetic field is due to a localized distribution of permanent magnetization, and in particular not to free currents, it satisfies the curl-free condition $\vec\nabla\times\vec H=0$. As a result, we may employ a magnetic scalar potential $\vec H=-\vec\nabla\Phi_M$. This allows us to write

$$\begin{aligned}\int \vec B\cdot\vec H\,d^3x = -\int \vec B\cdot\vec\nabla\Phi_M\,d^3x &= \int \Phi_M\vec\nabla\cdot\vec B\,d^3x - \int \vec\nabla\cdot(\Phi_M\vec B)\,d^3x\\ &= \int \Phi_M\vec\nabla\cdot\vec B\,d^3x - \int_\infty \Phi_M\vec B\cdot d\vec a\end{aligned}$$

where the surface term is taken over the 'surface' at infinity. So long as the distribution of magnetization is localized, the fields will fall off sufficiently fast at infinity so that the surface term vanishes. Finally, using Gauss' law for magnetism, $\vec\nabla\cdot\vec B=0$, gives the result

$$\int \vec B\cdot\vec H\,d^3x = 0 \tag{2}$$

Alternatively, the proof also follows from using the vector potential

$$\begin{aligned}\int \vec B\cdot\vec H\,d^3x = \int \vec H\cdot(\vec\nabla\times\vec A)\,d^3x &= \int \vec A\cdot(\vec\nabla\times\vec H)\,d^3x + \int \vec\nabla\cdot(\vec A\times\vec H)\,d^3x\\ &= \int \vec J\cdot\vec A\,d^3x + \int_\infty (\vec A\times\vec H)\cdot d\vec a\\ &= 0\end{aligned}$$

where we have dropped the term at infinity and used the fact that there are no free currents, $\vec J=0$.

b) From the potential energy (5.72) of a dipole in an external field, show that for a continuous distribution of permanent magnetization the magnetostatic energy can be written

$$W=\frac{\mu_0}{2}\int \vec H\cdot\vec H\,d^3x = -\frac{\mu_0}{2}\int \vec M\cdot\vec H\,d^3x$$

apart from an additive constant, which is independent of the orientation or position of the various constituent magnetized bodies.

The potential energy for a single dipole $\vec m$ in an external field $\vec B$ is given by

$$U=-\vec m\cdot\vec B$$

For a discrete distribution of point dipoles, the total magnetostatic energy is obtained by building up the configuration by starting with a single dipole, then bringing in the second one from infinity, then the third, and so on

$$W=-\sum_{i<j}\vec m_j\cdot\vec B_i$$

Here B_i is the magnetic induction caused by the i-th dipole

$$\vec{B}_i = \frac{\mu_0}{4\pi}\frac{3(\vec{m}_i \cdot \hat{x})\hat{x} - \vec{m}_i}{|\vec{x}\,|^3}$$

In particular, we note that $\vec{m}_j \cdot \vec{B}_i = \vec{m}_i \cdot \vec{B}_j$. As a result, we may write

$$W = -\frac{1}{2}\sum_{i\neq j} \vec{m}_j \cdot \vec{B}_i \tag{3}$$

This expression does not include any self-energy because we only take $i \neq j$.

For a continuous distribution of permanent magnetization, the generalization of (3) is evidently

$$W = -\frac{1}{2}\int \vec{M} \cdot \vec{B}\, d^3x \tag{4}$$

Unlike (3), however, this expression does include the self-energy. Nevertheless, this self-energy may be thought of as a constant which is independent of the orientation and position of the constituent magnetized bodies. Now, by writing $\vec{B} = \mu_0(\vec{H} + \vec{M})$, we may reexpress the energy as

$$W = -\frac{\mu_0}{2}\int \vec{M} \cdot (\vec{H} + \vec{M})\, d^3x = W_0 - \frac{\mu_0}{2}\int \vec{M} \cdot \vec{H}\, d^3x$$

where $W_0 = -(\mu_0/2)\int |\vec{M}|^2 d^3x$ is again an orientation and position independent constant related to the self-energy. Finally, using $\vec{M} = \frac{1}{\mu_0}\vec{B} - \vec{H}$ gives

$$W = W_0 - \frac{\mu_0}{2}\int \left(\frac{1}{\mu_0}\vec{B} - \vec{H}\right) \cdot \vec{H}\, d^3x = W_0 + \frac{\mu_0}{2}\int |\vec{H}|^2 d^3x$$

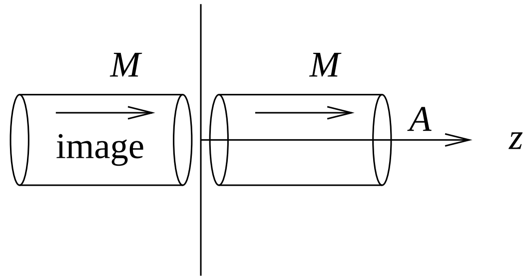
where we have used the result of part a, namely (2), to eliminate the $\int \vec{B} \cdot \vec{H}\, d^3x$ term.

5.22 Show that in general a long, straight bar of uniform cross-sectional area A with uniform lenghwise magnetization M, when placed with its flat end against an infinitely permeable flat surface, adheres with a force given approximately by

$$F \simeq \frac{\mu_0}{2} A M^2$$

Relate your discussion to the electrostatic considerations in Section 1.11.

This problem is best solved by considering an image magnet. The infinite permeability of the flat surface ensures that the magnetic field must be perpendicular to the surface. As a result, this is similar to the electrostatic case of electric field lines being perpendicular to the surface of a perfect conductor. For magnetostatics, this means that we may use a magnetic scalar potential Φ_M (since there are no free currents) subject to the condition $\Phi_M = 0$ at $z = 0$ (taking the surface to lie in the x-y plane at $z = 0$). The image problem is then set up as follows



Fortunately we may make use of some of our previous results. Since the force may be obtained by $\vec{F} = -\vec{\nabla} W$, we first compute the magnetostatic energy W. The previous problem has given various expressions for this energy. We choose to use (4)

$$W = -\frac{1}{2}\int \vec{M}\cdot\vec{B}\, d^3x$$

Here it is important to note that, while we solve this problem using an image magnet, the only quantities that show up in this energy integral are the actual sources of magnetization $\vec{M}$ and the actual magnetic induction $\vec{B}$. We place the magnet at a distance z_0 from the $z = 0$ surface so that

$$\vec{M}_{\text{real}} = \begin{cases} M\hat{z} & z_0 < z < z_0 + L \\ 0 & \text{otherwise} \end{cases}$$

As a result

$$W(z_0) = -\frac{M}{2}\int_{z_0}^{z_0+L} dz \int da\, B_z(\vec{x}\,) \approx -\frac{MA}{2}\int_{z_0}^{z_0+L} dz\, B_z(z) \tag{5}$$

where we have approximated that the magnetic induction is roughly uniform across the face of the magnet.

Using the image magnet setup, there are two sources of magnetic induction

$$\vec{B}(z) = \vec{B}_{\text{real}}(z) + \vec{B}_{\text{image}}(z)$$

Using (1) we see that

$$\vec{B}_{\text{real}}(z) = -\frac{\mu_0 M}{2}\left[\frac{z - z_0 - L}{\sqrt{a^2 + (z - z_0 - L)^2}} - \frac{z - z_0}{\sqrt{a^2 + (z - z_0)^2}}\right]\hat{z}$$

and

$$\vec{B}_{\text{image}}(z) = -\frac{\mu_0 M}{2}\left[\frac{z + z_0}{\sqrt{a^2 + (z + z_0)^2}} - \frac{z + z_0 + L}{\sqrt{a^2 + (z + z_0 + L)^2}}\right]\hat{z} \tag{6}$$

Here we have shifted the coordinates such that the real magnet lies between z_0 and z_0+L and the image magnet lies between $z=-z_0-L$ and $z=-z_0$. In principle, we may insert these expressions into (5) to compute the magnetostatic energy. However, as a simplification, we note that the integral of $\vec{M}\cdot\vec{B}_{\text{real}}$ gives a position independent (ie z_0 independent) self energy. Hence this will not contribute to the force. As a result, we only need to insert $\vec{B}_{\text{image}}$ into (5). This gives us

$$
\begin{aligned}
W(z_0) &\approx \frac{\mu_0 M^2 A}{4}\int_{z_0}^{z_0+L}\left[\frac{z+z_0}{\sqrt{a^2+(z+z_0)^2}}-\frac{z+z_0+L}{\sqrt{a^2+(z+z_0+L)^2}}\right]dz \\
&= \frac{\mu_0 M^2 A}{4}\left[\sqrt{a^2+(z+z_0)^2}-\sqrt{a^2+(z+z_0+L)^2}\right]_{z_0}^{z_0+L} \\
&= \frac{\mu_0 M^2 A}{4}\left[2\sqrt{a^2+4(z_0+L/2)^2}-\sqrt{a^2+4(z_0)^2}-\sqrt{a^2+4(z_0+L)^2}\right]
\end{aligned}
$$

The force is then

$$
\begin{aligned}
F_z = -\left.\frac{\partial W}{\partial z_0}\right|_{z_0=0} &\approx -\mu_0 M^2 A\left[\frac{2z_0+L}{\sqrt{a^2+4(z_0+L/2)^2}}-\frac{z_0}{\sqrt{a^2+4(z_0)^2}}\right. \\
&\qquad\left.-\frac{z_0+L}{\sqrt{a^2+4(z_0+L)^2}}\right]_{z_0=0} \\
&= -\mu_0 M^2 A\left[\frac{L}{\sqrt{a^2+L^2}}-\frac{L}{\sqrt{a^2+4L^2}}\right] \\
&\approx -\frac{\mu_0 M^2 A}{2}
\end{aligned}
$$

where in the last line we used $L\gg a$ (a condition that we needed anyway to ensure that B_z is nearly uniform on the endcaps).

Note that we could have alternatively used the result of Problem 5.20

$$
\vec{F} = -\int_V(\vec{\nabla}\cdot\vec{M})\vec{B}_e\,d^3x+\int_S(\vec{M}\cdot\vec{n}\,)\vec{B}_e\,da
$$

where the applied magnetic induction $\vec{B}_e$ is given by $\vec{B}_{\text{image}}$ in (6) with $z_0=0$. Since the magnetization is uniform, the force arises entirely from the surface term

$$
\begin{aligned}
\vec{F} &= \int_S(\vec{M}\cdot\hat{n})\vec{B}_e\,da = \hat{z}M\int[-B_z(0)+B_z(L)]\,da \\
&\approx \hat{z}MA[B_z(L)-B_z(0)] = \hat{z}\frac{\mu_0 M^2 A}{2}\left[\frac{2L}{\sqrt{a^2+4L^2}}-\frac{L}{\sqrt{a^2+L^2}}-\frac{L}{\sqrt{a^2+L^2}}\right] \\
&\approx -\hat{z}\frac{\mu_0 M^2 A}{2}
\end{aligned}
$$

What we have done here is to calculate the force through the magnetostatic energy

$$\vec{F} = -\vec{\nabla} W(\vec{x}\,)$$

where $\vec{x}$ denotes the position of the bar magnet. This is the magnetostatic equivalent of the force discussion in Section 1.11, which states that "Forces acting between charged bodies can be obtained by calculating the change in the total electrostatic energy of the system under small virtual displacements." In fact, this statement is true in general, provided we use the complete (electrostatic plus magnetostatic) energy of the system. Curiously, a conductor with surface-charge density σ feels an outward force of the form

$$F \approx \frac{\sigma^2 A}{2\epsilon_0}$$

which is roughly the electrostatic equivalent of

$$F \approx -\frac{\mu_0 M^2 A}{2}$$

found here.

5.27 A circuit consists of a long thin conducting shell of radius a and a parallel return wire of radius b on axis inside. If the current is assumed distributed uniformly throughout the cross section of the wire, calculate the self-inductance per unit length. What is the self-inductance if the inner conductor is a thin hollow tube?

For a uniformly distributed current I in a wire of radius b, the current density is given by

$$\vec{J} = \begin{cases} (I/\pi b^2)\hat{z} & \rho < b \\ 0 & \text{otherwise} \end{cases}$$

where we have taken the wire to be oriented along the z axis. By elementary application of Ampère's law, the magnetic induction is then

$$\vec{B} = \begin{cases} \dfrac{\mu I \rho}{2\pi b^2}\hat{\phi} & \rho < b \\ \dfrac{\mu_0 I}{2\pi\rho}\hat{\phi} & b < \rho < a \\ 0 & \rho > a \end{cases}$$

where we have allowed the wire to have a different permeability μ. The energy per unit length is then

$$\begin{aligned} W/\ell &= \frac{1}{2}\int_0^a \vec{B}\cdot\vec{H}\,2\pi\rho d\rho = \frac{I^2}{4\pi}\left[\mu\int_0^b \frac{\rho^3}{b^4}d\rho + \mu_0\int_b^a \frac{1}{\rho}d\rho\right] \\ &= \frac{I^2}{4\pi}\left[\frac{\mu}{4} + \mu_0 \log\frac{a}{b}\right] \end{aligned}$$

Setting the energy equal to $\frac{1}{2}LI^2$ gives an inductance per unit length of

$$L/\ell = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + \log\frac{a^2}{b^2}\right]$$

where $\mu_r = \mu/\mu_0$ is the relative permeability. Note that, if the inner conductor is a thin hollow tube, then all the current resides at $\rho = b$. In this case, the magnetic induction is

$$\vec{B} = \begin{cases} \dfrac{\mu_0 I}{2\pi\rho}\hat{\phi} & b < \rho < a \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$W/\ell = \frac{\mu_0 I^2}{4\pi}\log\frac{a}{b} \qquad \Rightarrow \qquad L/\ell = \frac{\mu_0}{4\pi}\log\frac{a^2}{b^2}$$