Physics 505 **Fall 2007**

Homework Assignment #8 — Solutions

Textbook problems: Ch. 5: 5.13, 5.14, 5.15, 5.16

5.13 A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

The charge density for a uniformly charged sphere of radius a is simply

$$\rho(\vec{x}\,) = \sigma\delta(|\vec{x}\,| - a)$$

Since the sphere is rotating with constant angular velocity $\vec{\omega}$, the velocity at any point $\vec{x}$ on the sphere is given by $\vec{v} = \vec{\omega} \times \vec{x}$. This allows us to write the current density as

$$\vec{J} = \rho\vec{v} = \sigma\,\vec{\omega} \times \vec{x}\,\delta(|\vec{x}\,| - a)$$

In Coulomb gauge, the vector potential is then given by

$$\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}d^3x' = \frac{\mu_0\sigma a^3}{4\pi}\vec{\omega} \times \int \frac{\hat{x}'}{|\vec{x} - \vec{x}\,'|}d\Omega' \tag{1}$$

where $|\vec{x}\,'| = a$. There are several ways to perform the angular integral. One quick method is to realize that the integral is a vector quantity. Then, by symmetry, once the $d\Omega'$ integral is performed, the only direction it can point in is given by $\hat{x}$. This allows us to write

$$\int \frac{\hat{x}'}{|\vec{x} - \vec{x}\,'|}d\Omega' = f(r)\hat{x}$$

where $f(r)$ is a function to be determined. In fact, by dotting both sides with $\hat{x}$, we see that

$$f = \int \frac{\cos\gamma}{|\vec{x} - \vec{x}\,'|}d\Omega' = \sum_l \frac{r_<^l}{r_>^{l+1}}\int P_1(\cos\gamma)P_l(\cos\gamma)d\Omega'$$

where $\cos\gamma = \hat{x}\cdot\hat{x}'$ and where

$$r_< = \min(r, a), \qquad r_> = \max(r, a)$$

Orthogonality of the Legendre polynomials then selects out $l = 1$, so that $f = (4\pi/3)(r_</r_>^2)$ or

$$\int \frac{\hat{x}'}{|\vec{x} - \vec{x}\,'|}d\Omega' = \frac{4\pi}{3}\frac{r_<}{r_>^2}\hat{x}$$

Inserting this into (1) gives

$$\vec{A}(\vec{x}\,) = \frac{\mu_0 \sigma a^3}{3r} \frac{r_<}{r_>^2} \vec{\omega} \times \vec{x}$$

More explicitly, we have

$$\vec{A} = \begin{cases} \dfrac{\mu_0 \sigma a}{3} \vec{\omega} \times \vec{x} & r < a \\ \dfrac{\mu_0 \sigma a^4}{3r^3} \vec{\omega} \times \vec{x} & r > a \end{cases}$$

The magnetic induction is now given by

$$\vec{B}_{\text{in}} = \vec{\nabla} \times \vec{A}_{\text{in}} = \frac{\mu_0 \sigma a}{3} \vec{\nabla} \times (\vec{\omega} \times \vec{x}\,) = \frac{2\mu_0 \sigma a}{3} \vec{\omega} \qquad r < a$$

and

$$\vec{B}_{\text{out}} = \vec{\nabla} \times \vec{A}_{\text{out}} = \frac{\mu_0 \sigma a^4}{3} \vec{\nabla} \times \left(\frac{\vec{\omega} \times \vec{x}}{r^3} \right) = \frac{\mu_0 \sigma a^4}{3} \frac{3\hat{x}(\omega \cdot \hat{x}) - \vec{\omega}}{r^3} \qquad r > a$$

The magnetic induction inside the sphere is uniform and parallel to the axis of rotation, while the magnetic induction outside is a dipole pattern with magnetic moment

$$\vec{m} = \frac{4\pi}{3} \sigma a^4 \vec{\omega}$$

5.14 A long, hollow, right circular cylinder of inner (outer) radius a (b), and of relative permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\vec{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $\vec{B}$ on the cylinder axis to $\vec{B}_0$ as a function of $\log_{10} \mu_r$ for $a^2/b^2 = 0.5$, 0.1. Neglect end effects.

For a long cylinder (neglecting end effects) we may think of this as a two-dimensional problem. Since there are no current sources, we use a magnetic scalar potential Φ_M which must be harmonic in two dimensions. Since $\vec{H} = -\vec{\nabla}\Phi_M$, we orient the uniform magnetic field H_0 along the $+x$ axis and write

$$\Phi_M(\rho, \phi) = \begin{cases} (-H_0 \rho + \frac{\alpha}{\rho}) \cos\phi, & \rho > b \\ (\beta\rho + \frac{\gamma}{\rho}) \cos\phi, & a < \rho < b \\ \delta\rho \cos\phi, & \rho < a \end{cases} \tag{2}$$

Of course, the general harmonic expansion would be of the form $(A_m \rho^m + B_m \rho^{-m}) \cos m\phi + (C_m \rho^m + D_m \rho^{-m}) \sin m\phi$. However here we have already used the shortcut that all matching conditions for $m \neq 1$ lead to homogeneous equations admitting only a trivial (zero) solution.

The magnetostatic boundary conditions demand that H_ϕ and B_ρ are continuous at both $\rho = a$ and $\rho = b$. This results in four equations for the four unknowns α, β, γ and δ. The magnetic field (and magnetic induction) components are

$$H_\phi = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi_M = \begin{cases} (-H_0 + \frac{\alpha}{\rho^2})\sin\phi, & \rho > b \\ (\beta + \frac{\gamma}{\rho^2})\sin\phi, & a < \rho < b \\ \delta\sin\phi, & \rho < a \end{cases}$$

and

$$B_\rho = -\mu\frac{\partial}{\partial\rho}\Phi_M = \begin{cases} \mu_0(H_0 + \frac{\alpha}{\rho^2})\cos\phi, & \rho > b \\ \mu(-\beta + \frac{\gamma}{\rho^2})\cos\phi, & a < \rho < b \\ -\mu_0\delta\cos\phi, & \rho < a \end{cases}$$

The resulting matching conditions at a and b are

$$-H_0 + \frac{\alpha}{b^2} = \beta + \frac{\gamma}{b^2}, \qquad H_0 + \frac{\alpha}{b^2} = \mu_r\left(-\beta + \frac{\gamma}{b^2}\right)$$
$$\beta + \frac{\gamma}{a^2} = \delta, \qquad \beta - \frac{\gamma}{a^2} = \frac{1}{\mu_r}\delta$$

where $\mu_r = \mu/\mu_0$. These equations may be solved to yield

$$\begin{aligned} \alpha &= \Delta^{-1}(\mu_r - \mu_r^{-1})(b^2 - a^2)H_0 \\ \beta &= -2\Delta^{-1}(1 + \mu_r^{-1})H_0 \\ \gamma &= -2\Delta^{-1}(1 - \mu_r^{-1})a^2 H_0 \\ \delta &= -4\Delta^{-1}H_0 \end{aligned}$$

where

$$\Delta = (1+\mu_r)(1+\mu_r^{-1}) + (1-\mu_r)(1-\mu_r^{-1})\left(\frac{a}{b}\right)^2 = \frac{1}{\mu_r}\left[(\mu_r+1)^2 - (\mu_r-1)^2\left(\frac{a}{b}\right)^2\right] \tag{3}$$

Substituting these coefficients in (2) gives the magnetic scalar potential

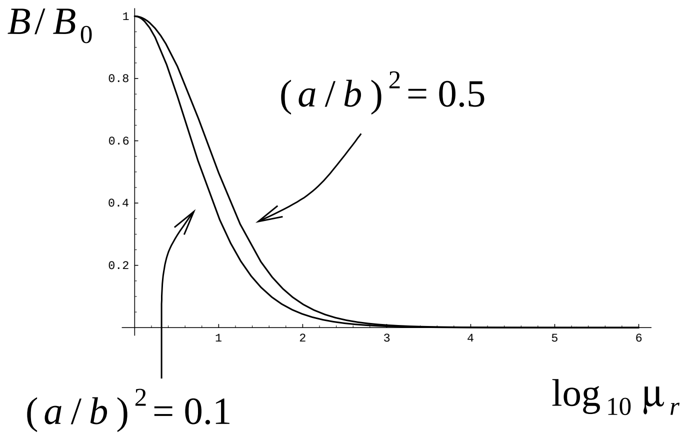
$$\Phi_M = -H_0\cos\phi \times \begin{cases} \rho - \dfrac{(b^2-a^2)(\mu_r - \mu_r^{-1})}{\Delta\rho}, & \rho > b \\ \dfrac{2}{\Delta}\left((1+\mu_r^{-1})\rho + (1-\mu_r^{-1})\dfrac{a^2}{\rho}\right), & a < \rho < b \\ \dfrac{4}{\Delta}\rho, & \rho < a \end{cases}$$

We see that the magnetic induction for $\rho < a$ is uniform, pointed along the same direction as $\vec{B}_0$, but reduced by a factor of $4/\Delta$. The other two regions contain a dipole field in addition to a uniform component.

Since $\vec{H} = -\vec{\nabla}\Phi_M = (4/\Delta)H_0\hat{x}$ for $\rho < a$, the ratio of $\vec{B}$ on axis ($\rho = 0$) to $\vec{B}_0$ is given by

$$\frac{B}{B_0} = \frac{4}{\Delta} = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2(a/b)^2} \tag{4}$$

This may be plotted as follows



5.15 Consider two long, straight wires, parallel to the z axis, spaced a distance d apart and carrying currents I in opposite directions. Describe the magnetic field $\vec{H}$ in terms of a magnetic scalar potential Φ_M, with $\vec{H} = -\vec{\nabla}\Phi_M$.

a) If the wires are parallel to the z axis with positions, $x = \pm d/2$, $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two-dimensional dipole

$$\Phi_M \approx -\frac{Id\sin\phi}{2\pi\rho} + \mathcal{O}(d^2/\rho^2)$$

where ρ and ϕ are the usual polar coordinates.

We start with the magnetic induction for a single wire located at $x = y = 0$ and carrying current in the $+\hat{z}$ direction. By elementary application of Ampère's law, we have

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} \qquad \Rightarrow \qquad \vec{H} = \frac{I}{2\pi\rho}\hat{\phi}$$

Assuming $\vec{H} = -\vec{\nabla}\Phi_M$, and using $\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}$ in polar coordinates, we see that the above magnetic field may be obtained from a magnetic scalar potential

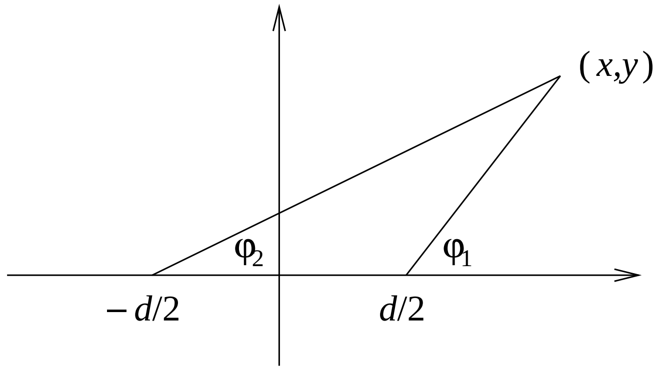
$$\Phi_M = -\frac{I}{2\pi}\phi \tag{5}$$

Note that this expression is multiple valued. The reason for this is that the current source at the origin (ie the wire carrying current I) violates the curl-free condition $\vec{\nabla} \times \vec{H} = 0$ at the origin. Although the magnetic scalar potential is multiple valued when we go around the origin, the physical magnetic field $\vec{H}$ is well defined everywhere (except at $\rho = 0$).

For two parallel wires with currents $\pm I$ (in the $\hat{z}$ direction) and positions $(x, y) = (\pm d/2, 0)$, we superpose (5) to obtain

$$\Phi_M = -\frac{I}{2\pi}(\phi_1 - \phi_2) = -\frac{I}{2\pi}\left(\tan^{-1}\frac{y}{x - d/2} - \tan^{-1}\frac{y}{x + d/2}\right)$$

where ϕ_1 and ϕ_2 are as indicated



Using

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\frac{\alpha-\beta}{1+\alpha\beta}$$

gives

$$\Phi_M = -\frac{I}{2\pi}\tan^{-1}\frac{yd}{x^2+y^2-d^2/4} = -\frac{I}{2\pi}\tan^{-1}\frac{(d/\rho)\sin\phi}{1-\frac{1}{4}(d/\rho)^2} \tag{6}$$

where we have used $\rho^2 = x^2+y^2$ and $y=\rho\sin\phi$. In the limit of small spacing, the arctan may be expanded in powers of d/ρ. The leading term gives the desired result

$$\Phi_M \approx -\frac{I}{2\pi}\frac{d}{\rho}\sin\phi + \mathcal{O}\left((d/\rho)^3\right) \tag{7}$$

Note that only odd powers of d/ρ appear in the Taylor expansion. In fact, the full expansion of (6) yields

$$\begin{aligned}\Phi_M &= -\frac{I}{2\pi}\left[\frac{d}{\rho}\sin\phi + \frac{1}{12}\left(\frac{d}{\rho}\right)^3\sin 3\phi + \frac{1}{80}\left(\frac{d}{\rho}\right)^5\sin 5\phi + \cdots\right] \\ &= -\frac{I}{2\pi}\sum_{n=0}^{\infty}\frac{2}{(2n+1)}\left(\frac{d}{2\rho}\right)^{2n+1}\sin[(2n+1)\phi]\end{aligned}$$

This form of $\Phi_M = \sum(A_m/\rho^m)\sin(m\phi)$ satisfies Laplace's equation in two-dimensional polar coordinates, as it must. The series converges for $\rho > d/2$, so it may be considered an 'outside' solution for this problem.

b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius a (b) and magnetic permeability $\mu = \mu_r\mu_0$. Determine the magnetic scalar potential in the three regions, $0<\rho<a$, $a<\rho<b$, and $\rho>b$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part a, but with a strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2b^2 - (\mu_r-1)^2a^2}$$

Relate your result to Problem 5.14.

Using the approximation (7), we write the magnetic scalar potential in all three regions as

$$\Phi_M = \begin{cases} \frac{\alpha}{\rho}\sin\phi, & \rho > b \\ (\beta\rho + \frac{\gamma}{\rho})\sin\phi, & a < \rho < b \\ (-\frac{I}{2\pi}\frac{d}{\rho} + \delta\rho)\sin\phi, & \rho < a \end{cases} \tag{8}$$

where α, β, γ and δ are constants to be determined. Note that we allow the dipole potential (7) in the interior region to be modified by the addition of $\delta\rho\sin\phi$ because of the presence of the steel cylinder. This expression for Φ_M is similar to that of (2) corresponding to a cylinder in a region of initially uniform magnetic induction $\vec{B}_0$. However, here the source term $-(I/2\pi)(d/\rho)\sin\phi$ is a dipole source for $\rho < a$ as opposed to a uniform source $-H_0\rho\cos\phi$ for $\rho > b$. In any case, we match H_ϕ and B_ρ at both $\rho = a$ and $\rho = b$. The components H_ϕ and B_ρ are

$$H_\phi = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi_M = \begin{cases} -\frac{\alpha}{\rho^2}\cos\phi, & \rho > b \\ -(\beta + \frac{\gamma}{\rho^2})\cos\phi, & a < \rho < b \\ (\frac{I}{2\pi}\frac{d}{\rho^2} - \delta)\cos\phi, & \rho < a \end{cases}$$

and

$$B_\rho = -\mu\frac{\partial}{\partial\rho}\Phi_M = \begin{cases} \mu_0\frac{\alpha}{\rho^2}\sin\phi, & \rho > b \\ \mu(-\beta + \frac{\gamma}{\rho^2})\sin\phi, & a < \rho < b \\ -\mu_0(\frac{I}{2\pi}\frac{d}{\rho^2} + \delta)\sin\phi, & \rho < a \end{cases}$$

The resulting matching conditions at a and b are

$$\frac{\alpha}{b^2} = \beta + \frac{\gamma}{b^2} \qquad \frac{\alpha}{b^2} = \mu_r\left(-\beta + \frac{\gamma}{b^2}\right)$$

$$\beta + \frac{\gamma}{a^2} = -\frac{I}{2\pi}\frac{d}{a^2} + \delta \qquad \beta - \frac{\gamma}{a^2} = \frac{1}{\mu_r}\left(\frac{I}{2\pi}\frac{d}{a^2} + \delta\right)$$

This simultaneous system of four equations for four unknowns may be solved to yield

$$\alpha = -\frac{2Id}{\pi\Delta}$$
$$\beta = -\frac{Id}{\pi\Delta}\frac{1-\mu_r^{-1}}{b^2}$$
$$\gamma = -\frac{Id}{\pi\Delta}(1+\mu_r^{-1})$$
$$\delta = -\frac{Id}{2\pi\Delta}\frac{(a^2-b^2)(\mu_r-\mu_r^{-1})}{a^2b^2}$$

where

$$\Delta = \frac{1}{\mu_r}\left[(\mu_r+1)^2 - (\mu_r-1)^2\left(\frac{a}{b}\right)^2\right]$$

is defined identically with (3). Substituting these coefficients into (8) gives

$$\Phi_M = -\frac{Id}{2\pi}\sin\phi \times \begin{cases} \dfrac{4}{\Delta\rho}, & \rho > b \\ \dfrac{2}{\Delta}\left(\dfrac{(1-\mu_r^{-1})}{b^2}\rho + \dfrac{(1+\mu_r^{-1})}{\rho}\right), & a < \rho < b \\ \dfrac{1}{\rho} - \dfrac{(b^2-a^2)(\mu_r-\mu_r^{-1})}{\Delta a^2 b^2}\rho, & \rho < a \end{cases}$$

Since the dipole field for $\rho < a$ is generated by the $1/\rho$ term in the third line above, we see that the external ($\rho > b$) field is also a dipole, but with strength reduced by the factor

$$F = \frac{4}{\Delta} = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2(a/b)^2} \tag{9}$$

Note that this shielding factor is identical to (4), which was obtained in the solution to Problem 5.14. This demonstrates that the shielding factor from inside to outside is identical to that from outside to inside.

c) Assuming that $\mu_r \gg 1$, and $b = a + t$, where the thickness $t \ll b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of steel at 20 G), $b = 1.25$ cm, $t = 3$ mm. The shielding effect is relevant for reduction of stray fields in residential and commercial 60 Hz, 110 or 220 V wiring. [The figure illustrates the shielding effect for $a/b = 0.9$, $\mu_r = 100$.]

Taking $a = b - t$, we may express the ratio $a/b = 1 - t/b$. Substituting this into (9) gives

$$F = \frac{4\mu_r}{4\mu_r + (t/b)(2-t/b)(\mu_r-1)^2} = \left[1 + \mu_r\frac{t}{2b}\left(1 - \frac{t}{2b}\right)(1-\mu_r^{-1})^2\right]^{-1}$$
$$\approx \left(1 + \frac{\mu_r t}{2b}\right)^{-1}$$

Substituting in the above values gives

$$F \approx \frac{1}{25} = 0.04$$

In other words, the external dipole field is reduced by a factor of 25 compared with the field inside the steel shield.

5.16 A circular loop of wire of radius a and negligible thickness carries a current I. The loop is centered in a spherical cavity of radius $b > a$ in a large block of soft iron. Assume that the relative permeability of the iron is effectively infinite and that of the medium in the cavity, unity.

$a)$ In the approximation of $b \gg a$, show that the magnetic field at the center of the loop is augmented by a factor $(1 + a^3/2b^3)$ by the presence of the iron.

Since this problem involves a current loop, we take a vector potential approach. For a circular loop of radius a in the x-y plane in free space, we have found the expression for the vector potential

$$A_\phi^{\text{loop}} = \frac{\mu_0 I a}{2} \sum_{l \text{ odd}} \frac{1}{l(l+1)} \frac{r_<^l}{r_>^{l+1}} P_l^1(0) P_l^1(\cos\theta)$$

where $r_< = \min(r, a)$ and $r_> = \max(r, a)$. Of course, since the current loop is centered in a spherical cavity, this cannot be the complete answer. In order to find the appropriate solution, we note that, by linear superposition, we may add an arbitrary solution to Laplace's equation, $\nabla^2 \vec{A} = 0$, to the above. Taking advantage of cylindrical symmetry, the general solution to the homogeneous equation may be written as

$$A_\phi^{(0)} = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l^1(\cos\theta)$$

where A_l and B_l are coefficients to be determined. (This was essentially demonstrated in Problem 5.8.) Taking an inside solution requires us to set $B_l = 0$, so that the vector potential is well behaved as $r \to 0$. Superposing $\vec{A}^{\text{loop}}$ and $\vec{A}^{(0)}$ then gives

$$A_\phi = \frac{\mu_0 I a}{2} \sum_{l \text{ odd}} \frac{1}{l(l+1)} \left(\frac{r_<^l}{r_>^{l+1}} + \alpha_l r^l \right) P_l^1(0) P_l^1(\cos\theta) \tag{10}$$

where we have chosen to write the homogeneous solution using a new set of expansion coefficients α_l for convenience. Although the homogeneous solution could in principle involve even l terms, it is easy to see that such terms must vanish as they would not be sourced by the current loop (which only generates odd l terms).

Assuming the relative permeability of the iron is effectively infinite, the boundary conditions indicate that the magnetic field must end up being perpendicular to the surface of the cavity. In other words, the parallel component of $\vec{B}$ must vanish at $r = b$. Because of azimuthal symmetry, the only parallel component of interest is

$$\begin{aligned}
B_\theta(r = b) &= -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \bigg|_{r=b} \\
&= \frac{\mu_0 I a}{2} \sum_{l \text{ odd}} \frac{1}{l(l+1)} \left(l \frac{a^l}{r^{l+2}} - (l+1) \alpha_l r^{l-1} \right) P_l^1(0) P_l^1(\cos\theta) \bigg|_{r=b} \\
&= \frac{\mu_0 I a}{2} \sum_{l \text{ odd}} \frac{b^{l-1}}{l(l+1)} \left(l \frac{a^l}{b^{2l+1}} - (l+1) \alpha_l \right) P_l^1(0) P_l^1(\cos\theta)
\end{aligned}$$

where we have used $r_< = a$ and $r_> = r$ when computing B_θ at the surface of the cavity. Setting this parallel component to zero gives

$$\alpha_l = \frac{l}{l+1}\frac{a^l}{b^{2l+1}}$$

in which case (10) becomes

$$A_\phi = \frac{\mu_0 I a}{2}\sum_{l \text{ odd}} \frac{1}{l(l+1)}\left(\frac{r_<^l}{r_>^{l+1}} + \frac{l}{l+1}\frac{(ar)^l}{b^{2l+1}}\right) P_l^1(0)P_l^1(\cos\theta)$$

Note that for $r < a$ this expression may be rewritten as

$$A_\phi = \frac{\mu_0 I}{2}\sum_{l \text{ odd}} \frac{1}{l(l+1)}\left(\frac{r}{a}\right)^l \left(1 + \frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1}\right) P_l^1(0)P_l^1(\cos\theta)$$

Using

$$\frac{d}{dx}[\sqrt{1-x^2}P_l^1(x)] = l(l+1)P_l(x)$$

gives

$$\begin{aligned} B_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) \\ &= -\frac{\mu_0 I}{2a}\sum_{l \text{ odd}} \left(\frac{r}{a}\right)^{l-1}\left(1 + \frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1}\right) P_l^1(0)P_l(\cos\theta) \end{aligned}$$

and

$$\begin{aligned} B_\theta &= -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \\ &= -\frac{\mu_0 I}{2a}\sum_{l \text{ odd}} \frac{1}{l}\left(\frac{r}{a}\right)^{l-1}\left(1 + \frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1}\right) P_l^1(0)P_l^1(\cos\theta) \end{aligned}$$

This demonstrates that the l-th component of the magnetic induction is enhanced by a factor

$$F_l = 1 + \frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1} \tag{11}$$

compared to a current loop in free space. The field at the center of the loop is given by setting $r = 0$, in which case only the $l = 1$ term contributes. The enhancement factor is then

$$F_1 = 1 + \frac{1}{2}\left(\frac{a}{b}\right)^3 \tag{12}$$

Note that this result is valid, even if b is not much greater than a (so long as we assume the iron has infinite relative permeability).

b) What is the radius of the "image" current loop (carrying the same current) that simulates the effect of the iron for $r < b$?

As long as we work in the limit $b \gg a$, the $l = 1$ enhancement factor F_1 dominates over the others in the sense that $(a/b)^{2l+1}$ gets much smaller as l increases. Because of this, all we need out of an approximate "image loop' is the correct enhancement of the magnetic dipole. In the dipole approximation, an isolated current loop of radius R has magnetic induction

$$B_z = \frac{\mu_0 I}{2R}$$

at its center. For the enhancement factor F_1 given in (12), the magnetic induction is explicitly

$$B_z = \frac{\mu_0 I}{2a}\left(1 + \frac{1}{2}\left(\frac{a}{b}\right)^3\right) = \frac{\mu_0 I}{2a} + \frac{\mu_0 I}{2(2b^3/a^2)}$$

The first term on the right is interpreted as the magnetic induction from the real loop, while the second is that from the image loop. This indicates that the radius of the image loop is

$$R_{\text{image}} = \frac{2b^3}{a^2}$$

which is greater than b (as appropriate for an image).

Note that no exact single image loop solution is possible, as the enhancement factor (11) is l dependent, and a l dependent radius simply does not make sense. The reason this approximation makes sense for $b \gg a$ is that in this case the $l > 1$ image moments are insignificant, and may be ignored.