Physics 505 **Fall 2007**

Homework Assignment #7 — Solutions

Textbook problems: Ch. 5: 5.4, 5.7, 5.8, 5.9

5.4 A magnetic induction $\vec{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho,z)$ and $B_\rho(\rho,z)$ and with a known $B_z(0,z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

a) Show that near the axis the axial and radial components of magnetic induction are approximately

$$B_z(\rho,z) \approx B_z(0,z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots$$
$$B_\rho(\rho,z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots$$

Near the axis ($\rho \approx 0$), we may perform a series expansion of the magnetic induction

$$\begin{aligned} B_z(\rho,z) &= b_0(z) + \rho b_1(z) + \tfrac{1}{2}\rho^2 b_2(z) + \cdots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z) \\ B_\rho(\rho,z) &= c_0(z) + \rho c_1(z) + \tfrac{1}{2}\rho^2 c_2(z) + \cdots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z) \\ B_\phi(\rho,z) &= 0 \end{aligned} \tag{1}$$

Since the region is current free, the magnetic induction must satisfy the magnetostatic equations $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{B} = 0$. In cylindrical coordinates, the divergence equation gives

$$0 = \vec{\nabla}\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho + \frac{\partial}{\partial z}B_z = \sum_{n=0}^{\infty}\left(\frac{(n+1)\rho^{n-1}}{n!}c_n(z) + \frac{\rho^n}{n!}b_n'(z)\right)$$

We shift $n \to n+1$ in the first term in the sum to obtain

$$0 = \frac{1}{\rho}c_0(z) + \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left(\frac{n+2}{n+1}c_{n+1}(z) + b_n'(z)\right)$$

Since this vanishes for any value of ρ, each term in the sum must vanish individually. This gives us

$$c_0(z) = 0, \qquad c_{n+1} = -\frac{n+1}{n+2}b_n'(z) \tag{2}$$

We now continue with the curl equation. The only non-trivial component is

$$0 = [\vec{\nabla} \times \vec{B}]_\phi = \frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z = \sum_{n=0}^{\infty} \left(\frac{\rho^n}{n!} c'_n(z) - \frac{\rho^{n-1}}{(n-1)!} b_n(z) \right)$$

Shifting $n \to n+1$ in the second term in the sum gives

$$0 = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \left(c'_n(z) - b_{n+1}(z) \right)$$

so that

$$b_{n+1}(z) = c'_n(z)$$

Combining this with (2) gives us the recursion relation

$$b_{n+1}(z) = c'_n(z) = -\frac{n}{n+1} b''_{n-1}(z)$$

Taking into account the fact that $c_0(z) = 0$, we see that $b_1(z) = 0$. On the other hand, $b_0(z)$ is undetermined. The recursion relation can then be used to relate $b_n(z)$ to $b_0(z)$ where n is an even integer. The result is

$$\begin{aligned} b_n(z) &= (-1)^{n/2} \frac{(n-1)(n-3)\cdots 3 \cdot 1}{n(n-2)\cdots 4 \cdot 2} b_0^{(n)}(z) \\ &= \frac{(-1)^{n/2}}{2^n} \frac{n!}{[(n/2)!]^2} b_0^{(n)}(z) \qquad (n \text{ even}) \end{aligned} \tag{3}$$

where $b_0^{(n)}(z)$ indicates the n-th derivative of $b_0(z)$. Using (2), we then see that the odd $c_n(z)$ terms are non-vanishing, and are given by

$$c_{n+1}(z) = \frac{(-1)^{n/2+1}}{2^n} \frac{(n+1)!}{(n+2)[(n/2)!]^2} b_0^{(n+1)}(z) \qquad (n \text{ odd})$$

Inserting these expressions into the Taylor expansion (1) finally gives

$$\begin{aligned} B_z(\rho, z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \frac{\rho^{2k}}{(k!)^2} \left[\frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}} \right] \\ B_\rho(\rho, z) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k} \frac{\rho^{(2k+1)}}{(2k+2)(k!)^2} \left[\frac{\partial^{2k+1} B_z(0,z)}{\partial z^{2k+1}} \right] \end{aligned} \tag{4}$$

where we have made the identification $b_0(z) = B_z(0, z)$. The first few terms in the expansion gives the desired result

$$\begin{aligned} B_z(\rho, z) &= B_z(0,z) - \frac{\rho^2}{4} \left[\frac{\partial^2 B_z(0,z)}{\partial z^2} \right] + \cdots \\ B_\rho(\rho, z) &= -\frac{\rho}{2} \left[\frac{\partial B_z(0,z)}{\partial z} \right] + \frac{\rho^3}{16} \left[\frac{\partial^3 B_z(0,z)}{\partial z^3} \right] + \cdots \end{aligned} \tag{5}$$

Incidentally, another approach to this problem is to note that, for a current-free region, the magnetic induction may be given in terms of a magnetic scalar potential

$$\vec{B} = -\vec{\nabla}\Phi_M, \qquad \nabla^2\Phi_M = 0$$

Solving Laplace's equation in cylindrical coordinates gives an expansion in terms of Bessel functions. Choosing the modes to be oscillating in ρ and exponential in z allows us to write

$$\Phi_M(\rho, \phi, z) = \sum_m \int dk\, A_m(k) J_m(k\rho) e^{im\phi} e^{kz}$$

where $A_m(k)$ are a set of expansion coefficients. Since the problem is cylindrically symmetric, only the $m = 0$ component contributes

$$\Phi_M(\rho, z) = \int dk\, A_0(k) J_0(k\rho) e^{kz}$$

As long as we are only interested in the field near the axis, we may Taylor expand the Bessel function $J_0(k\rho)$ for small ρ. Using

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} \tag{6}$$

we find

$$\Phi_M(\rho, z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \int dk\, A_0(k) k^{2s} e^{kz}$$

Since the exponential e^{kz} blows up as $z \to \infty$ (for positive k) or $z \to \infty$ (for negative k), this expression is not be well defined in all of space. However, we may consider either working in a finite range of z or treating this as a formal expression. In any case, we note that powers of k inside the integral corresponds to taking z derivatives of the exponential

$$k \quad \leftrightarrow \quad \frac{\partial}{\partial z} \tag{7}$$

Hence

$$\Phi_M(\rho, z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \int dk\, A_0(k) e^{kz}$$

Setting $\rho = 0$ in this expression gives

$$\Phi_M(0, z) = \int dk\, A_0(k) e^{kz}$$

Substituting this back in allows us to write

$$\Phi_M(0,z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \Phi_M(0,z)$$

Finally, using $\vec{B} = -\vec{\nabla}\Phi_M$ allows us to reproduce the series expansion of the magnetic induction given in (4). Incidentally, we note that the coefficients in the solution to the $b_n(z)$ recursion relation given in (3) matches the Taylor coefficients of the J_0 Bessel function, (6). In fact, if desired, we may use (7) to write the formal expression

$$\Phi_M(\rho,z) = J_0\left(\rho\frac{\partial}{\partial z}\right)\Phi_M(0,z)$$

Then

$$\begin{aligned} B_z(\rho,z) &= J_0\left(\rho\frac{\partial}{\partial z}\right) B_z(0,z) \\ B_\rho(\rho,z) &= J_0'\left(\rho\frac{\partial}{\partial z}\right) B_z(0,z) = -J_1\left(\rho\frac{\partial}{\partial z}\right) B_z(0,z) \end{aligned}$$

b) What are the magnitudes of the neglected terms, or equivalently what is the criterion defining "near" the axis?

We see from (4) that the n-th term in the series expansion of $\vec{B}(\rho,z)$ is of the form

$$\sim \frac{\rho^n}{2^n[(n/2)!]^2}\left[\frac{\partial^n B_z(0,z)}{\partial z^n}\right]$$

Ignoring constant factors, this indicates that the ratio of adjacent terms in the series is roughly

$$\frac{a_{n+2}}{a_n} \sim \rho^2 \frac{[\partial^{n+2} B_z(0,z)/\partial z^{n+2}]}{[\partial^n B_z(0,z)/\partial z^n]}$$

In general, this ratio needs to be less than one for the series to converge. Hence, this provides a criterion for being "near" the axis

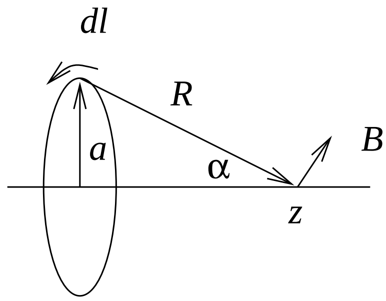
$$\rho \ll \sqrt{\frac{[\partial^n B_z(0,z)/\partial z^n]}{[\partial^{n+2} B_z(0,z)/\partial z^{n+2}]}}$$

For a smooth function $B_z(0,z)$, the n-th derivative behaves roughly as $1/L^n$ where L is the scale of variation of the field. As a result, we demand $\rho \ll L$ where L is a typical length for the variation of the magnetic induction B_z along the z direction.

5.7 A compact circular coil of radius a, carrying a current I (perhaps N turns, each with current I/N), lies in the x-y plane with its center at the origin.

a) By elementary means [Eq. (5.4)] find the magnetic induction at any point on the z axis

As long as we restrict ourselves to the z axis, the magnetic induction is given by an elementary application of the Biot-Savart law.



By symmetry, only the z component contributes

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{[d\vec{\ell}\times\vec{R}]_z}{R^3} = \frac{\mu_0 I}{4\pi}\int\frac{d\ell\, R\sin\alpha}{R^3} = \frac{\mu_0 I}{4\pi}2\pi a\frac{a}{R^3} = \frac{\mu_0 I a^2}{2R^3}$$

Substituting in $R^2 = a^2 + z^2$ yields

$$B_z = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$$

b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above, the first coil. With the coordinate origin relocated at the point midway between the centers of the two coils, determine the magnetic induction on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8} + \cdots\right]$$

where $d^2 = a^2 + b^2/4$.

By shifting the origin around, it should be obvious that the magnetic induction is given exactly by

$$B_z = \frac{\mu_0 I a^2}{2}\left((a^2+(z-\tfrac{1}{2}b)^2)^{-3/2} + (a^2+(z+\tfrac{1}{2}b)^2)^{-3/2}\right) \tag{8}$$

All we must do now is to Taylor expand the terms to order z^4. Noting that we are seeking an expansion in powers of z/d^2, we may write

$$\begin{aligned} B_z &= \frac{\mu_0 I a^2}{2}\left((d^2 - bz + z^2)^{-3/2} + (d^2+bz+z^2)^{-3/2}\right) \\ &= \frac{\mu_0 I a^2}{2d^3}\left((1 - b\zeta + d^2\zeta^2)^{-3/2} + (1+b\zeta+d^2\zeta^2)^{-3/2}\right) \\ &= \frac{\mu_0 I a^2}{2d^3}\left((1 - b\zeta + (a^2+\tfrac{1}{4}b^2)\zeta^2)^{-3/2} + (1+b\zeta+(a^2+\tfrac{1}{4}b^2)\zeta^2)^{-3/2}\right) \end{aligned} \tag{9}$$

where we have introduced $\zeta = z/d^2$. Expanding this in powers of ζ yields

$$B_z = \frac{\mu_0 I a^2}{2d^3}\left[1 + \tfrac{3}{2}(b^2 - a^2)\zeta^2 + \tfrac{15}{16}(b^4 - 6b^2a^2 + 2a^4)\zeta^4 + \cdots\right] \tag{10}$$

which is the desired result.

c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form

$$B_z = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right); \qquad B_\rho = -\sigma_2 z\rho$$

In principle, we may compute the vector potential or magnetic induction off-axis through the Biot-Savart law. However, near the axis, it is more convenient to perform a series expansion of the magnetic induction and use the results of problem 5.4 above. The result of part b indicates that

$$B_z(0, z) = \sigma_0 + \sigma_2 z^2 + \cdots$$

where

$$\sigma_0 = \frac{\mu_0 I a^2}{d^3}, \qquad \sigma_1 = \frac{3(b^2 - a^2)}{2d^4}\sigma_0$$

Inserting this expansion into (5) gives

$$\begin{aligned}
B_z(\rho, z) &= [\sigma_0 + \sigma_2 z^2 + \cdots] - \tfrac{1}{4}\rho^2[\sigma_0 + \sigma_2 z^2 + \cdots]'' + \cdots \\
&= \sigma_0 + \sigma_2(z^2 - \tfrac{1}{2}\rho^2) + \cdots \\
B_\rho(\rho, z) &= -\tfrac{1}{2}\rho[\sigma_0 + \sigma_2 z^2 + \cdots]' + \cdots \\
&= -\sigma_2 \rho z + \cdots
\end{aligned}$$

d) For the two coils in part b show that the magnetic induction on the z axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part b by the formal substitution $d \to |z|$.

For large $|z|$ we Taylor expand (8) in inverse powers of z

$$B_z = \frac{\mu_0 I a^2}{2|z|^3}\Big((1 - bz^{-1} + (a^2 + \tfrac{1}{4}b^2)z^{-2})^{-3/2} + (1 + bz^{-1} + (a^2 + \tfrac{1}{4}b^2)z^{-2})^{-3/2}\Big)$$

Comparing this with the last line of (9) shows that the Taylor series is formally equivalent under the substitution $\zeta \to z^{-1}$, which may be accomplished by taking $d \to |z|$.

e) If $b = a$, the two coils are known as a pair of Helmholtz coils. For this choice of geometry the second terms in the expansions of parts b and d are absent ($\sigma_2 = 0$

in part c). The field near the origin is then very uniform. What is the maximum permitted value of $|z|/a$ if the axial field is to be uniform to one part in 10^4, one part in 10^2?

For $b = a$ the axial field is of the form

$$B_z = \frac{\mu_0 I a^2}{2d^3}\left(1 - \frac{45}{16}\frac{a^4 z^4}{d^8} + \cdots\right)$$
$$= \frac{4\mu_0 I a^2}{5^{3/2}a^3}\left(1 - \frac{144}{125}\left(\frac{z}{a}\right)^4 + \cdots\right)$$

Taking the $(|z|/a)^4$ term as a small correction, the field non-uniformity is

$$\frac{\delta B}{B} \approx \frac{144}{125}\left(\frac{z}{a}\right)^4$$

For uniformity to one part in 10^4, we find $|z|/a < 0.097$, while for uniformity to one part in 10^2, we instead obtain $|z|/a < 0.305$. These numbers are actually pretty good because of the fourth power. For example, the first value indicates we can move $\approx \pm 10\%$ of the distance between the coils while maintaining field uniformity at the level of 0.01%. Helmholtz coils are very useful in the lab for canceling out the Earth's magnetic field.

5.8 A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\vec{J} = \hat{\phi} J(r,\theta)$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside.

a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L \mu_L r^{-L-1} P_L^1(\cos\theta)$$

outside the current distribution.

b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int d^3x\, r^{-L-1} P_L^1(\cos\theta) J(r,\theta)$$

and

$$\mu_L = -\frac{1}{L(L+1)}\int d^3x\, r^L P_L^1(\cos\theta) J(r,\theta)$$

We work out both parts a and b simultaneously. For a current density $\vec{J}$, the vector potential in Coulomb gauge has the expression

$$\begin{aligned}\vec{A}(\vec{x}\,) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}d^3x' \\ &= \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r',\theta',\phi')Y_l^m(\theta,\phi)Y_l^{m\,*}(\theta',\phi')\,r'^2dr'd\phi'd(\cos\theta')\end{aligned}$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. For the cylindrically symmetric current distribution $\vec{J} = \hat{\phi}J(r,\theta)$, the vector potential becomes

$$\begin{aligned}\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\hat{\phi}'J(r',\theta') \\ \times Y_l^m(\theta,0)Y_l^{m\,*}(\theta',0)e^{im(\phi-\phi')}\,r'^2dr'd\phi'd(\cos\theta')\end{aligned}$$

where we have also explicitly written out the azimuthal dependence of the spherical harmonics. Noting that $Y_l^{m\,*}(\theta,\phi) = (-1)^m Y_l^{-m}(\theta,\phi)$ and that $Y_l^m(\theta,0) = Y_l^{m\,*}(\theta,0)$, the sum over m may be put into the form

$$\begin{aligned}\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\hat{\phi}'J(r',\theta') \\ \times Y_l^m(\theta,0)Y_l^{m\,*}(\theta',0)\cos[m(\phi-\phi')]\,r'^2dr'd\phi'd(\cos\theta')\end{aligned} \tag{11}$$

We now focus on the azimuthal part of this integral

$$I_m \equiv \int_0^{2\pi}\hat{\phi}'\cos[m(\phi-\phi')]d\phi' \tag{12}$$

paying attention to the fact that the primed $\hat{\phi}'$ direction may not be the same as the unprimed $\hat{\phi}$ direction for the observer. In particular, we note that the cartesian components of the spherical coordinate basis vectors are

$$\begin{aligned}\hat{r} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \hat{\theta} &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \hat{\phi} &= (-\sin\phi, \cos\phi, 0)\end{aligned}$$

Thus an arbitrary vector $\vec{V}$ may be decomposed as

$$\vec{V} = (\hat{r}\cdot\vec{V})\hat{r} + (\hat{\theta}\cdot\vec{V})\hat{\theta} + (\hat{\phi}\cdot\vec{V})\hat{\phi}$$

In particular, for the $\hat{\phi}'$ unit vector, this decomposition becomes

$$\hat{\phi}' = \sin(\phi-\phi')(\hat{r}\sin\theta + \hat{\theta}\cos\theta) + \cos(\phi-\phi')\hat{\phi}$$

Substituting this into (12) yields

$$
\begin{aligned}
I_m &= \int_0^{2\pi} \left[\sin(\phi-\phi')(\hat{r}\sin\theta + \hat{\theta}\cos\theta) + \cos(\phi-\phi')\hat{\phi}\right] \cos[m(\phi-\phi')]d\phi' \\
&= \pi[\delta_{m,1} + \delta_{m,-1}]\hat{\phi}
\end{aligned}
$$

As a result, the expression for the vector potential, (11), becomes

$$
\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\sum_l \frac{4\pi}{2l+1} 2\pi\hat{\phi} \int \frac{r_<^l}{r_>^{l+1}} J(r',\theta') Y_l^m(\theta,0) Y_l^{m\,*}(\theta',0)\, r'^2 dr' d(\cos\theta')
$$

This demonstrates that only the azimuthal component of the vector potential is non-vanishing (in Coulomb gauge). Writing out the spherical harmonics in terms of associated Legendre polynomials gives

$$
\begin{aligned}
A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\sum_l \frac{P_l^1(\cos\theta)}{l(l+1)} 2\pi \int \frac{r_<^l}{r_>^{l+1}} J(r',\theta') P_l^1(\cos\theta')\, r'^2 dr' d(\cos\theta') \\
&= \frac{\mu_0}{4\pi}\sum_l \frac{P_l^1(\cos\theta)}{l(l+1)} \int \frac{r_<^l}{r_>^{l+1}} J(r',\theta') P_l^1(\cos\theta')\, d^3x'
\end{aligned}
$$

For the interior region, we take $r < r'$, since the observer at r is closer to the origin than then region containing the current $J(r',\theta')$. This gives

$$
A_\phi^{\text{in}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{r^l}{l(l+1)} P_l^1(\cos\theta) \int \frac{J(r',\theta')}{r'^{l+1}} P_l^1(\cos\theta')\, d^3x'
$$

On the other hand, $r_<$ and $r_>$ are flipped for the exterior region ($r > r'$), so that

$$
A_\phi^{\text{out}}(r,\theta) = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)r^{l+1}} P_l^1(\cos\theta) \int r'^l J(r',\theta') P_l^1(\cos\theta')\, d^3x'
$$

Finally, defining

$$
\begin{aligned}
m_l &= -\frac{1}{l(l+1)} \int \frac{P_l^1(\cos\theta)}{r^{l+1}} J(r,\theta)\, d^3x \\
\mu_l &= -\frac{1}{l(l+1)} \int r^l P_l^1(\cos\theta) J(r,\theta)\, d^3x
\end{aligned}
\tag{13}
$$

gives the desired vector potential multipole expansion

$$
\begin{aligned}
A_\phi^{\text{in}} &= -\frac{\mu_0}{4\pi}\sum_l m_l r^l P_l^1(\cos\theta) \\
A_\phi^{\text{out}} &= -\frac{\mu_0}{4\pi}\sum_l \frac{\mu_l}{r^{l+1}} P_l^1(\cos\theta)
\end{aligned}
\tag{14}
$$

Note that an alternate method of solving part a is to the vector potential in a current-free region satisfies the homogeneous equation

$$\nabla^2 \vec{A} = 0 \qquad \text{(in Coulomb gauge where } \vec{\nabla} \cdot \vec{A} = 0\text{)}$$

Taking $\vec{A} = A_\phi(r,\theta)\hat{\phi}$ and using the expression for the Laplacian in spherical coordinates gives

$$0 = \left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](A_\phi(r,\theta)\hat{\phi})$$

Note that the Laplacian must also act on the unit vector $\hat{\phi}$, as it rotates along with the azimuthal angle ϕ. In particular, it is easy to see that

$$\frac{\partial^2}{\partial\phi^2}\hat{\phi} = -\hat{\phi}$$

This results in the Laplace's equation

$$0 = \left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\right]A_\phi(r,\theta)$$

which may be compared with the ordinary separation of variables solution to the scalar Laplacian in spherical coordinates. Since this corresponds to an azimuthal quantum number $m = \pm 1$, the solution is then of the form

$$A_\phi(r,\theta) = \sum_l [A_l r^l + B^l r^{-l-1}]P_l^1(\cos\theta)$$

(where we have used the fact that P_l^1 and P_l^{-1} are linearly dependent, so we are free to restrict to $m = 1$). Taking the appropriate inside and outside solutions, then immediately gives (14).

5.9 The two circular coils of radius a and separation b of Problem 5.7 can be described in cylindrical coordinates by the curent density

$$\vec{J} = \hat{\phi}I\delta(\rho - a)[\delta(z - b/2) + \delta(z + b/2)]$$

a) Using the formalism of Problem 5.8, calculate the internal and external multipole moments for $L = 1, \ldots, 5$.

In principle, all we are asked to do here is to compute the moments (13) using the current density given above. However, there may be a bit of a subtlety in whether we choose to use cylindrical or spherical coordinates. Although (13) was worked out in spherical coordinates, the integral can be performed in any coordinate system, so long as we use the appropriate expressions for r and $\cos\theta$. Since $\vec{J}$ is given in cylindrical coordinates, we take this to be the natural coordinate system to use. In this case, (13) may be reexpressed as

$$\begin{aligned} m_l &= -\frac{1}{l(l+1)} \int \frac{P_l^1(z/\sqrt{\rho^2+z^2})}{(\rho^2+z^2)^{(l+1)/2}} J(\rho,z)\, \rho d\rho d\phi dz \\ \mu &= -\frac{1}{l(l+1)} \int (\rho^2+z^2)^{l/2} P_l^1(z/\sqrt{\rho^2+z^2}) J(\rho,z)\, \rho d\rho d\phi dz \end{aligned}$$

where we have made use of the transformation

$$r = \sqrt{\rho^2+z^2}, \qquad \tan\theta = \frac{\rho}{z}$$

Using

$$J = I\delta(\rho-a)[\delta(z-b/2)+\delta(z+b/2)]$$

immediately gives us

$$\begin{aligned} m_l &= -\frac{2\pi I a}{l(l+1)} d^{-l-1}[P_l^1(b/2d) + P_l^1(-b/2d)] \\ \mu_l &= -\frac{2\pi I a}{l(l+1)} d^{l}[P_l^1(b/2d) + P_l^1(-b/2d)] \end{aligned}$$

where we have defined

$$d = \sqrt{a^2+b^2/4}$$

Since the associated Legendre polynomials have definite parity

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x)$$

we see that only the odd l moments survive

$$\begin{aligned} m_l &= -\frac{4\pi I a}{l(l+1)} d^{-l-1} P_l^1(b/2d) \\ \mu_l &= -\frac{4\pi I a}{l(l+1)} d^{l} P_l^1(b/2d) \end{aligned}$$

Since

$$\begin{aligned} P_1^1(x) &= -\sqrt{1-x^2} \\ P_3^1(x) &= -\tfrac{3}{2}(5x^2-1)\sqrt{1-x^2} \\ P_5^1(x) &= -\tfrac{15}{8}(21x^4-14x^2+1)\sqrt{1-x^2} \end{aligned} \tag{15}$$

the first few internal moments are

$$\begin{aligned} m_1 &= \frac{2\pi I a^2}{d^3} \\ m_3 &= \frac{2\pi I a^2}{d^5}\frac{b^2-a^2}{4d^2} \\ m_5 &= \frac{2\pi I a^2}{d^7}\frac{b^4-6a^2b^2+2a^4}{16d^4} \end{aligned} \tag{16}$$

and the first few external moments are

$$\mu_1 = 2\pi I a^2$$
$$\mu_3 = 2\pi I a^2 \frac{b^2 - a^2}{4}$$
$$\mu_5 = 2\pi I a^2 \frac{b^4 - 6a^2b^2 + 2a^4}{16}$$

b) Using the internal multipole expansion of Problem 5.8, write down explicitly an expression for B_z on the z axis and relate it to the answer of Problem 5.7b.

We begin with the internal multipole expansion of the vector potential, which was given in (14)

$$\begin{aligned} A_\phi &= -\frac{\mu_0}{4\pi}\left[m_1 r P_1^1(\cos\theta) + m_3 r^3 P_3^1(\cos\theta) + m_5 r^5 P_5^1(\cos\theta) + \cdots\right] \\ &= \frac{\mu_0}{4\pi}\sin\theta\big[m_1 r + m_3 r^3 \tfrac{3}{2}(5\cos^2\theta - 1) \\ &\qquad\qquad + m_5 r^5 \tfrac{15}{8}(21\cos^4\theta - 14\cos^2\theta + 1) + \cdots\big] \end{aligned} \tag{17}$$

Note that we have used the explicit forms of the associated Legendre polynomials given in (15). Since A_ϕ is the only non-vanishing component of $\vec{A}$, the magnetic induction is given by

$$\vec{B} = \vec{\nabla}\times(A_\phi\hat{\phi}) = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

Substituting in (17) gives

$$\begin{aligned} \vec{B} = \frac{\mu_0}{2\pi}\Big[&\hat{r}\cos\theta\Big(m_1 + 3m_3r^2(5\cos^2\theta - 3) \\ &\qquad + \tfrac{15}{8}m_5r^4(63\cos^4\theta - 70\cos^2\theta + 15) + \cdots\Big) \\ &- \hat{\theta}\sin\theta\Big(m_1 + 3m_3r^2(5\cos^2\theta - 1) \\ &\qquad + \tfrac{45}{8}m_5r^4(21\cos^4\theta - 14\cos^2\theta + 1) + \cdots\Big)\Big] \end{aligned}$$

To obtain the magnetic induction on the z axis, we let $\theta = 0$ in the above and find

$$B_z = \frac{\mu_0}{2\pi}\left(m_1 + 6m_3z^2 + 15m_5z^4 + \cdots\right)$$

(where we have taken $r = z$ along the axis). Finally, substituting in the internal moments, (16), gives

$$B_z = \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3}{2}\frac{b^2 - a^2}{d^2}\left(\frac{z}{d}\right)^2 + \frac{15}{16}\frac{b^4 - 6a^2b^2 + 2a^4}{d^4}\left(\frac{z}{d}\right)^4 + \cdots\right]$$

which agrees with the answer to Problem 5.7b, given by (10).