Physics 505 **Fall 2007**

Homework Assignment #7 — Due Thursday, November 1

Textbook problems: Ch. 5: 5.4, 5.7, 5.8, 5.9

5.4 A magnetic induction $\vec{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ and with a known $B_z(0, z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

a) Show that near the axis the axial and radial components of magnetic induction are approximately

$$B_z(\rho, z) \approx B_z(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0, z)}{\partial z^2}\right] + \cdots$$
$$B_\rho(\rho, z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0, z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0, z)}{\partial z^3}\right] + \cdots$$

b) What are the magnitudes of the neglected terms, or equivalently what is the criterion defining "near" the axis?

5.7 A compact circular coil of radius a, carrying a current I (perhaps N turns, each with current I/N), lies in the x-y plane with its center at the origin.

a) By elementary means [Eq. (5.4)] find the magnetic induction at any point on the z axis

b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above, the first coil. With the coordinate origin relocated at the point midway between the centers of the two coils, determine the magnetic induction on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2 - a^2)z^2}{2d^4} + \frac{15(b^4 - 6b^2a^2 + 2a^4)z^4}{16d^8} + \cdots\right]$$

where $d^2 = a^2 + b^2/4$.

c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form

$$B_z = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right); \qquad B_\rho = -\sigma_2 z\rho$$

d) For the two coils in part b show that the magnetic induction on the z axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part b by the formal substitution $d \to |z|$.

e) If $b = a$, the two coils are known as a pair of Helmholtz coils. For this choice of geometry the second terms in the expansions of parts b and d are absent ($\sigma_2 = 0$ in part c). The field near the origin is then very uniform. What is the maximum permitted value of $|z|/a$ if the axial field is to be uniform to one part in 10^4, one part in 10^2?

5.8 A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\vec{J} = \hat{\phi} J(r, \theta)$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside.

a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r, \theta) = -\frac{\mu_0}{4\pi} \sum_L m_L r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r, \theta) = -\frac{\mu_0}{4\pi} \sum_L \mu_L r^{-L-1} P_L^1(\cos\theta)$$

outside the current distribution.

b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)} \int d^3x\, r^{-L-1} P_L^1(\cos\theta) J(r, \theta)$$

and

$$\mu_L = -\frac{1}{L(L+1)} \int d^3x\, r^L P_L^1(\cos\theta) J(r, \theta)$$

5.9 The two circular coils of radius a and separation b of Problem 5.7 can be described in cylindrical coordinates by the curent density

$$\vec{J} = \hat{\phi} I \delta(\rho - a)[\delta(z - b/2) + \delta(z + b/2)]$$

a) Using the formalism of Problem 5.8, calculate the internal and external multipole moments for $L = 1, \ldots, 5$.

b) Using the internal multipole expansion of Problem 5.8, write down explicitly an expression for B_z on the z axis and relate it to the answer of Problem 5.7b.