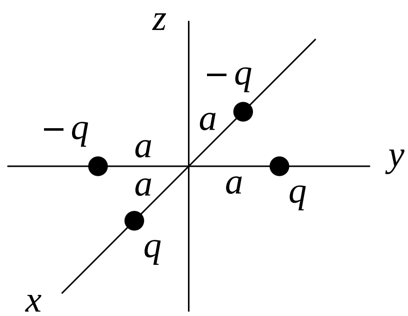
Physics 505 **Fall 2007**

Homework Assignment #6 — Due Thursday, October 25

Textbook problems: Ch. 4: 4.1, 4.7, 4.8, 4.9

4.1 Calculate the multipole moments q_{lm} of the charge distributions shown as parts *a)* and *b)*. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.

a)



The multipole moments are given by

$$q_{lm} = \int \rho(\vec{x}) r^l Y^*_{lm}(\theta,\phi)\, d^3x = qa^l[Y^*_{lm}(\tfrac{\pi}{2},0)+Y^*_{lm}(\tfrac{\pi}{2},\tfrac{\pi}{2})-Y^*_{lm}(\tfrac{\pi}{2},\pi)-Y^*_{lm}(\tfrac{\pi}{2},\tfrac{3\pi}{2})]$$

This is given in terms of associated Legendre polynomials by

$$q_{lm} = qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)[1+(-i)^m-(-1)^m-(i)^m]$$

The moments vanish unless m is odd. Writing $m=2k+1$ gives

$$\begin{aligned} q_{l,2k+1} &= 2qa^l[1-i(-1)^k]\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1))!}{(l+(2k+1))!}}P_l^{2k+1}(0) \\ &= 2qa^l[1-i(-1)^k]Y_{l,2k+1}(\tfrac{\pi}{2},0) \end{aligned}$$

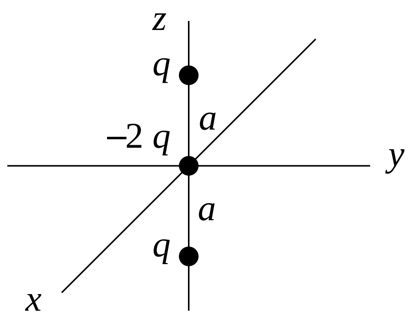
Note that by parity this vanishes unless l is odd. Hence only the odd l and m moments are present. The lowest non-trivial ones are

$$q_{1,1} = -q^*_{1,-1} = -2qa(1-i)\sqrt{\frac{3}{8\pi}}$$

and

$$q_{3,3} = -q^*_{3,-3} = -2qa^3(1+i)\frac{1}{4}\sqrt{\frac{35}{4\pi}} \qquad q_{3,1} = -q^*_{3,-1} = 2qa^3(1-i)\frac{1}{4}\sqrt{\frac{21}{4\pi}}$$

$b)$



In this case, we have

$$q_{lm} = qa^l[Y^*_{lm}(0,0) + Y^*_{lm}(\pi,0)]$$

for $l > 0$ and $q_{00} = 0$. By azimuthal symmetry, only the $m = 0$ moments are non-vanishing. Hence

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{4\pi}}[P_l(1) + P_l(-1)] = qa^l[1 + (-1)^l]\sqrt{\frac{2l+1}{4\pi}} \qquad l > 0$$

We end up with even multipoles

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{\pi}} \qquad l = 2, 4, 6, \ldots$$

Explicitly

$$q_{20} = qa^2\sqrt{\frac{5}{\pi}} \qquad q_{40} = qa^4\sqrt{\frac{9}{\pi}}$$

$c)$ For the charge distribution of the second set $b)$ write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.

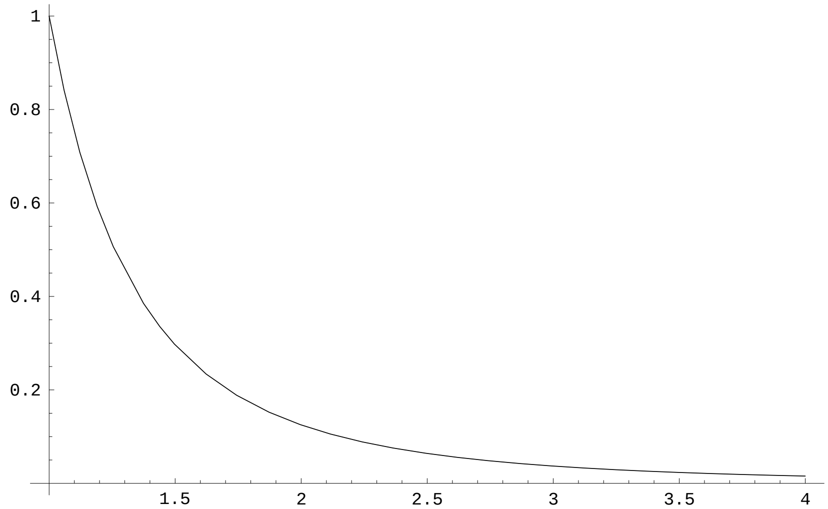
The expansion of the potential is

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} = \frac{1}{\epsilon_0}\sum_{l=2,4,\ldots}\frac{qa^l}{2l+1}\sqrt{\frac{2l+1}{\pi}}\frac{Y_{l0}(\theta,\phi)}{r^{l+1}} \\ &= \frac{q}{2\pi\epsilon_0}\sum_{l=2,4,\ldots}\frac{a^l}{r^{l+1}}P_l(\cos\theta) = \frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}(3\cos^2\theta - 1) + \cdots\end{aligned}$$

In the x-y plane we have $\cos\theta = 0$, so the lowest order term is

$$\Phi = -\frac{q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3 + \cdots$$

We all know what $1/r^3$ looks like when plotted, but here it is

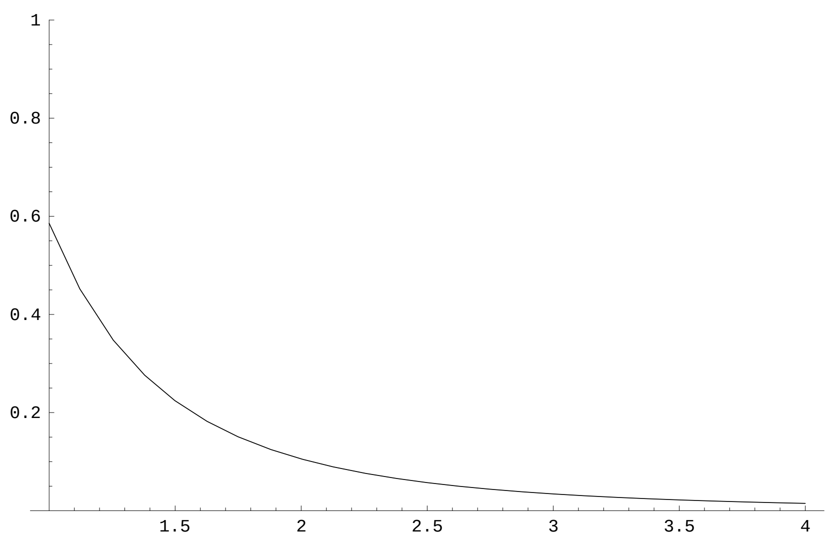


d) Calculate directly from Coulomb's law the exact potential for *b*) in the *x*-*y* plane. Plot it as a function of distance and compare with the result found in part *c*).

For three charges, the potential is simply the sum of three terms, one for each charge. In the *x*-*y* plane, if r is the distance from the origin we have

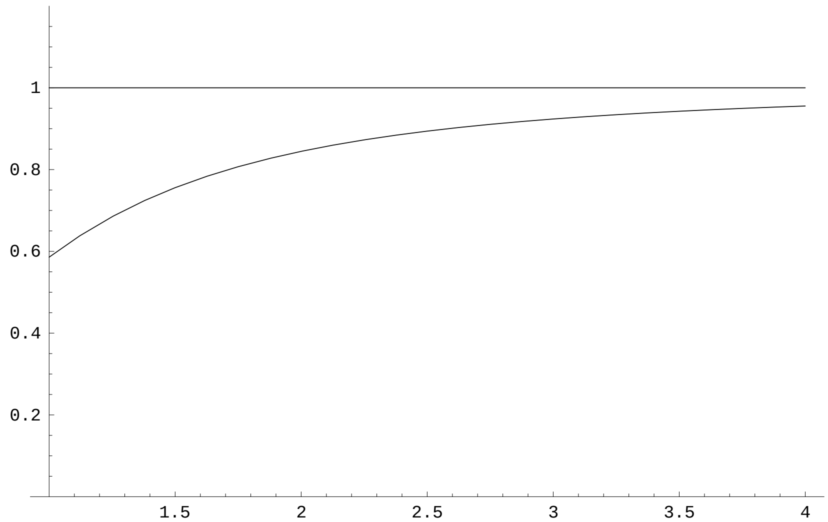
$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r} + \frac{1}{\sqrt{r^2+a^2}}\right) = -\frac{q}{2\pi\epsilon_0 r}\left(1 - \frac{1}{\sqrt{1+(a/r)^2}}\right)$$
$$= -\frac{q}{4\pi\epsilon_0 a}2\left(\frac{1}{(r/a)} - \frac{1}{\sqrt{1+(r/a)^2}}\right)$$

The exact potential looks like



Divide out the asymptotic form in parts *c*) and *d*) to see the behavior at large distances more clearly.

If we divide out by $1/r^3$, the approximate and exact potentials are



where the straight line is the approximation of c) and the sloped line is the exact result. The approximation improves as $r \gg a$.

4.7 A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

This charge distribution is azimuthally symmetric. As a result, only $m = 0$ moments will be nonvanishing. Furthermore, noting that

$$\sin^2\theta = 1 - \cos^2\theta = \tfrac{2}{3}[P_0(\cos\theta) - P_2(\cos\theta)]$$

we may write down the moments

$$\begin{aligned}
q_{l0} &= \int \rho(r,\theta) r^l Y_{l0}^*(\theta,\phi)\, r^2\, dr\, d\phi\, d(\cos\theta) \\
&= 2\pi\sqrt{\frac{2l+1}{4\pi}} \int \rho(r,\theta) r^l P_l(\cos\theta)\, r^2\, dr\, d(\cos\theta) \\
&= \frac{2\pi}{64\pi}\frac{2}{3}\sqrt{\frac{2l+1}{4\pi}} \int_0^\infty r^{l+4} e^{-r}\, dr \int_{-1}^1 P_l(\cos\theta)[P_0(\cos\theta) - P_2(\cos\theta)]\, d(\cos\theta) \\
&= \frac{1}{48}\sqrt{\frac{2l+1}{4\pi}}\Gamma(l+5)[2\delta_{l,0} - \tfrac{2}{5}\delta_{l,2}]
\end{aligned}$$

As a result, we read off the only nonvanishing multipole moments

$$q_{00} = \sqrt{\frac{1}{4\pi}}, \qquad q_{20} = -6\sqrt{\frac{5}{4\pi}}$$

The multipole expansion then yields the large distance potential

$$\begin{aligned}\Phi &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &= \frac{1}{4\pi\epsilon_0}\sum_{l}\sqrt{\frac{4\pi}{2l+1}}q_{l0}\frac{P_l(\cos\theta)}{r^{l+1}} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}-\frac{6}{r^3}P_2(\cos\theta)\right]\end{aligned} \tag{1}$$

b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4}-\frac{r^2}{120}P_2(\cos\theta)\right]$$

We may use a Green's function to obtain the potential at any point in space. In general (since there are no boundaries, except at infinity)

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

However, for azimuthal symmetry, it is sufficient to focus on the $m=0$ terms in the expansion

$$G(\vec{x},\vec{x}\,') = \sum_{l}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)P_l(\cos\theta') + (m\neq 0)$$

Then

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int\rho(\vec{x}\,')G(\vec{x},\vec{x}\,')\,d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\pi}{64\pi}\frac{2}{3}\int_0^\infty r'^4e^{-r'}\frac{r_<^l}{r_>^{l+1}}\,dr' \\ &\quad\times\int_{-1}^{1}[P_0(\cos\theta')-P_2(\cos\theta')]P_l(\cos\theta')P_l(\cos\theta)\,d(\cos\theta') \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{48}\left[\frac{1}{r^{l+1}}\int_0^r r'^{l+4}e^{-r'}\,dr' + r^l\int_r^\infty r'^{3-l}e^{-r'}\,dr'\right][2\delta_{l,0}-\tfrac{2}{5}\delta_{l,2}P_2(\cos\theta)]\end{aligned}$$

Instead of writing this out in terms of incomplete Gamma functions, it is better just to integrate for $l=0$ and $l=2$. The result is

$$\begin{aligned}\Phi = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\bigg[&\frac{1}{r}\left(24-e^{-r}(24+18r+6r^2+r^3)\right) \\ &-\frac{1}{r^3}P_2(\cos\theta)\left(144-e^{-r}(144+144r+72r^2+24r^3+6r^4+r^5)\right)\bigg]\end{aligned}$$

Note that as $r \to \infty$ the e^{-r} factors are exponentially small. As a result, we simply reproduce (1) in this limit. On the other hand, as $r \to 0$, a Taylor expansion yields

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\left(\frac{1}{4} + \cdots\right) - \left(\frac{r^2}{120} + \cdots\right) P_2(\cos\theta)\right] \tag{2}$$

Obtaining the correct $l = 2$ term involves the cancellation of the first five terms in the Taylor expansion. Note that the leading terms in the final expression have the 'correct' powers of $r^l P_l(\cos\theta)$ in order to satisfy Laplace's equation.

c) If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}$ m^2, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(\vec{r})$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}$ m. Express your answer as a frequency by dividing by Planck's constant h.

The charge density in this problem is that for the $m = \pm 1$ states of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

We first note that if we put the correct units of electronic charge e and Bohr radius a_0 into the charge distribution ρ, the potential near the origin (2) becomes

$$\Phi = -\frac{e}{4\pi\epsilon_0 a_0}\left[\frac{1}{4} - \frac{1}{120}\left(\frac{r}{a_0}\right)^2 P_2(\cos\theta) + \cdots\right]$$

where the overall minus sign is due to the negative charge of the electron. (We take $e > 0$). The interaction energy is then

$$W = \int \rho_N \Phi\, d^3x = -\frac{e}{4\pi\epsilon_0 a_0}\int \rho_N \left[\frac{1}{4} - \frac{1}{120}\left(\frac{r}{a_0}\right)^2 P_2(\cos\theta) + \cdots\right] d^3x$$

where ρ_N is the charge density of the nucleus. Since $\int \rho_N d^3x = Ze$ gives the total charge of the nucleus, we write

$$W = -\frac{e^2}{4\pi\epsilon_0 a_0}\left[\frac{Z}{4} - \frac{1}{240a_0^2}\frac{1}{e}\int \rho_N r^2(3\cos^2\theta - 1)d^3x + \cdots\right]$$

where we have used $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Using $z = r\cos\theta$, this may be rewritten as

$$\begin{aligned} W &= -\frac{e^2}{4\pi\epsilon_0 a_0}\left[\frac{Z}{4} - \frac{1}{240a_0^2}\frac{1}{e}\int \rho_N (3z^2 - r^2)d^3x + \cdots\right] \\ &= -\frac{e^2}{4\pi\epsilon_0 a_0}\left[\frac{Z}{4} - \frac{Q}{240a_0^2} + \cdots\right] \end{aligned}$$

where we have used the (classical) definition of the nuclear quadrupole moment

$$Q = \frac{1}{e}\int \rho_N(3z^2 - r^2)d^3x$$

The first term is the electrostatic interaction energy. The quadrupole interaction energy (expressed as a frequency) is

$$W/h = \frac{e^2}{4\pi\epsilon_0\hbar c}\frac{Qc}{480\pi a_0^3} = \frac{\alpha Qc}{480\pi a_0^3} \approx 1\,\mathrm{MHz}$$

where $\alpha \approx 1/137.036$ is the fine structure constant, and where we have put in the numerical value of Q. This nuclear quadrupole interaction with the electric field of the electron cloud typically gives rise to radio frequency resonances (in the low megahertz range) that may be detected using the process of nuclear quadrupole resonance (NQR). Since NQR is sensitive to the electronic structure (ie chemical bonds), it has seen some application towards explosives detection. In particular, nitrogen is a common element in many explosives, and since ^{14}N has a non-zero quadrupole moment, NQR can be used to detect what sorts of nitrogen compounds may be present in a sample.

4.8 A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

a) Determine the potential and electric field in the three regions, neglecting end effects.

Since the cylinder is very long, we treat this as a two-dimensional problem. In this case, the potential admits a general expansion

$$\Phi = \sum_m [\alpha_m\rho^m + \beta_m\rho^{-m}]\cos(m\phi - \delta_m)$$

(where the $m = 0$ term should actually be $\alpha_0 + \beta_0 \log\rho$). Furthermore, by orienting the electric field along the $+x$ direction, we may use the $\phi \leftrightarrow -\phi$ symmetry of this problem to eliminate the phases δ_m. As a result, we are able to write the potential as an expansion in each of the three regions

$$\Phi = \begin{cases} \Phi_1 = A_0 + \sum_m A_m\rho^{-m}\cos(m\phi) - E_0\rho\cos\phi, & \rho > b \\ \Phi_2 = B_0 + C_0\log\rho + \sum_m (B_m\rho^m + C_m\rho^{-m})\cos(m\phi), & a < \rho < b \\ \Phi_3 = D_0 + \sum_m D_m\rho^m\cos(m\phi), & \rho < a \end{cases}$$

For each value of m, there are four unknowns, A_m, B_m, C_m and D_m. On the other hand, there are also four matching conditions ($D^\perp$ and $E^\parallel$ both at a and at b). Note, however, that when $m \neq 1$ these matching conditions yield homogeneous equations which only admit the trivial solution

$$A_m = B_m = C_m = D_m = 0 \qquad m \neq 1$$

(Although the $m = 0$ case has to be treated separately, it is easy to see that $C_0 = 0$. The remaining constants must satisfy $A_0 = B_0 = D_0$, and may be taken to vanish, since an overall constant added to the potential is unphysical.) Thus we may focus on $m = 1$ and write

$$\Phi = \begin{cases} \Phi_1 = (A\rho^{-1} - E_0\rho)\cos\phi, & \rho > b \\ \Phi_2 = (B\rho + C\rho^{-1})\cos\phi, & a < \rho < b \\ \Phi_3 = D\rho\cos\phi, & \rho < a \end{cases} \tag{3}$$

We may obtain the electric field by taking a gradient

$$\begin{aligned} E_\rho &= -\frac{\partial\Phi}{\partial\rho} = \begin{cases} E_\rho^1 = (A\rho^{-2} + E_0)\cos\phi, & \rho > b \\ E_\rho^2 = (-B + C\rho^{-2})\cos\phi, & a < \rho < b \\ E_\rho^3 = -D\cos\phi, & \rho < a \end{cases} \\ E_\phi &= -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = \begin{cases} E_\phi^1 = (A\rho^{-2} - E_0)\sin\phi, & \rho > b \\ E_\phi^2 = (B + C\rho^{-2})\sin\phi, & a < \rho < b \\ E_\phi^3 = D\sin\phi, & \rho < a \end{cases} \end{aligned} \tag{4}$$

The matching at $\rho = a$ is

$$\epsilon_0 E_\rho^3 = \epsilon E_\rho^2\Big|_{\rho=a}, \qquad E_\phi^3 = E_\phi^2\Big|_{\rho=a}$$

or

$$(\epsilon_0/\epsilon)D - B + Ca^{-2} = 0, \qquad D - B - Ca^{-2} = 0$$

This may be solved for C and D in terms of B

$$C = \frac{1 - \epsilon_0/\epsilon}{1 + \epsilon_0/\epsilon}Ba^2, \qquad D = \frac{2}{1 + \epsilon_0/\epsilon}B \tag{5}$$

Similarly, the matching at $\rho = b$ is

$$\epsilon E_\rho^2 = \epsilon_0 E_\rho^1\Big|_{\rho=b}, \qquad E_\phi^2 = E_\phi^1\Big|_{\rho=b}$$

or

$$(\epsilon_0/\epsilon)Ab^{-2} + B - Cb^{-2} = -(\epsilon_0/\epsilon)E_0, \qquad Ab^{-2} - B - Cb^{-2} = E_0$$

Eliminating C using (5) gives rise to the simultaneous equations

$$\begin{pmatrix} b^{-2} & -1 - \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}\left(\frac{a}{b}\right)^2 \\ (\epsilon_0/\epsilon)b^{-2} & 1 - \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}\left(\frac{a}{b}\right)^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = E_0 \begin{pmatrix} 1 \\ -\epsilon_0/\epsilon \end{pmatrix}$$

This yields a solution

$$\begin{aligned} A &= E_0 \Delta^{-1}(1 - \epsilon_0/\epsilon)\left(1 - \left(\frac{a}{b}\right)^2\right) b^2 \\ B &= -E_0 \Delta^{-1}(2\epsilon_0/\epsilon) \end{aligned}$$

where

$$\Delta = (1 + \epsilon_0/\epsilon)\left(1 - \left(\frac{1 - \epsilon_0/\epsilon}{1 + \epsilon_0/\epsilon}\frac{a}{b}\right)^2\right)$$

is b^2 times the determinant of the above matrix. Substituting B into (5) then gives the remaining coefficients

$$\begin{aligned} C &= -E_0 \Delta^{-1} \frac{(1 - \epsilon_0/\epsilon)2\epsilon_0/\epsilon}{1 + \epsilon_0/\epsilon} a^2 \\ D &= -E_0 \Delta^{-1} \frac{4\epsilon_0/\epsilon}{1 + \epsilon_0/\epsilon} \end{aligned}$$

These expressions may be simplified to read

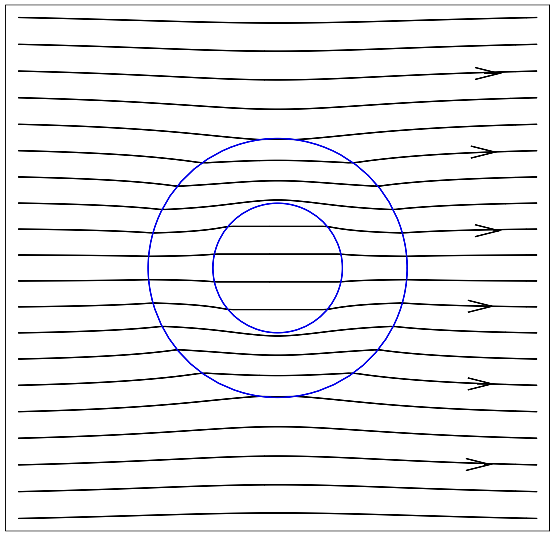
$$\begin{aligned} A &= E_0 b^2 \frac{(\epsilon^2 - \epsilon_0^2)(b^2 - a^2)}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \\ B &= -2E_0 \frac{\epsilon_0(\epsilon + \epsilon_0)b^2}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \\ C &= -2E_0 a^2 \frac{\epsilon_0(\epsilon - \epsilon_0)b^2}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \\ D &= -4E_0 \frac{\epsilon\epsilon_0 b^2}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \end{aligned} \tag{6}$$

The potential and electric field are obtained by substituting these coefficients into (3) and (4). For the potential, we have

$$\begin{aligned} b < \rho: &\qquad \Phi_1 = E_0 \left[\frac{(\epsilon^2 - \epsilon_0^2)(b^2 - a^2)}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \frac{b^2}{\rho} - \rho \right] \cos\phi \\ a < \rho < b: &\qquad \Phi_2 = -2E_0 \frac{\epsilon_0 b^2[(\epsilon + \epsilon_0)\rho + (\epsilon - \epsilon_0)a^2/\rho]}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \cos\phi \\ \rho < a: &\qquad \Phi_3 = -4E_0 \frac{\epsilon\epsilon_0 b^2 \rho}{(\epsilon + \epsilon_0)^2 b^2 - (\epsilon - \epsilon_0)^2 a^2} \cos\phi \end{aligned} \tag{7}$$

b) Sketch the lines of force for a typical case of $b \simeq 2a$.

For $\epsilon/\epsilon_0 = 1.5$, the 'electric field' lines look like



Note that we have actually plotted the electric displacement field $\vec{D}$, as Gauss' law in vacuum $\vec{\nabla} \cdot \vec{D} = 0$ ensures that the lines of electric displacement are continuous and unbroken. The electric field lines themselves are discontinuous at the interface between dielectrics.

c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

A solid dielectric cylinder of radius b may be obtained by taking the limit $a \to 0$. In this case the expressions (6) and (7) simplify considerably. We give the potential

$$\Phi = \begin{cases} \Phi_1 = -E_0 x + E_0 \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} \frac{b^2 x}{\rho^2}, & \rho > b \\ \Phi_2 = -E_0 \frac{2\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} x, & \rho < b \end{cases} \tag{8}$$

where $x = \rho\cos\phi$. The potential Φ_3 is irrelevant in this case. Here we see that the potential Φ_2 inside the cylinder is uniform (but corresponds to a reduced electric field provided $\epsilon > \epsilon_0$). The potential outside is that of the original uniform electric field combined with a two-dimensional dipole.

For the opposite limit, we obtain a cylindrical cavity of radius a by taking the limit $b \to \infty$. In this case, we end up with

$$\Phi = \begin{cases} \Phi_2 = -E_0 \frac{2\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} x - E_0 \frac{2\epsilon_0/\epsilon(1-\epsilon_0/\epsilon)}{(1+\epsilon_0/\epsilon)^2} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -E_0 \frac{4\epsilon_0/\epsilon}{(1+\epsilon_0/\epsilon)^2} x, & \rho < a \end{cases}$$

At first glance, this appears to be considerably different from (8). However, note that the physical electric field we measure as $\rho \to \infty$ is $\tilde{E}_0 = E_0(2\epsilon_0/\epsilon)/(1+\epsilon_0/\epsilon)$.

In terms of $\tilde{E}_0$, we have

$$\Phi = \begin{cases} \Phi_2 = -\tilde{E}_0 x - \tilde{E}_0 \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -\tilde{E}_0 \frac{2}{1+\epsilon_0/\epsilon} x, & \rho < a \end{cases}$$

which may be rewritten as

$$\Phi = \begin{cases} \Phi_2 = -\tilde{E}_0 x + \tilde{E}_0 \frac{1-\epsilon/\epsilon_0}{1+\epsilon/\epsilon_0} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -\tilde{E}_0 \frac{2\epsilon/\epsilon_0}{1+\epsilon/\epsilon_0} x, & \rho < a \end{cases}$$

This agrees with (8) after the replacement $\epsilon \leftrightarrow \epsilon_0$ (and $a \to b$), as it must.

4.9 A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0.

a) Find the potential at all points in space as an expansion in spherical harmonics.

By symmetry, we may place the point charge on the z-axis at $z = d$. In this case, the problem is azimuthally symmetric, and we may expand the potential in Legendre polynomials instead of spherical harmonics. For the potential inside the dielectric sphere, we take

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon} \sum_l \alpha_l \left(\frac{r}{a}\right)^l P_l(\cos\theta) \tag{9}$$

where the $q/4\pi\epsilon$ prefactor is taken for convenience (but can be absorbed into a redefinition of α_l if so desired). Note that we do not need any source term, since there are no charges inside the sphere. On the other hand, the solution outside the sphere is given by

$$\Phi_{\text{out}} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{x} - d\hat{z}|} + \Phi_0$$

where Φ_0 is a homogeneous solution to Laplace's equation, $\nabla^2\Phi_0 = 0$. Expanding in Legendre polynomials allows us to write

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0} \sum_l \left[\frac{r_<^l}{r_>^{l+1}} + \beta_l \left(\frac{a}{r}\right)^{l+1} \right] P_l(\cos\theta) \tag{10}$$

Note that $r_< = \min(r, d)$ and $r_> = \max(r, d)$. Since we must match the parallel electric field and perpendicular electric displacement at $r = a$, we may take $r_< = r$ and $r_> = d$ when using Φ_{out} in the matching equations. For the parallel electric field, we have

$$E_\theta^{\text{in}} = -\frac{1}{r} \left.\frac{\partial\Phi_{\text{in}}}{\partial\theta}\right|_{r=a} = \frac{q}{4\pi\epsilon} \sum_l \frac{\alpha_l}{a} P_l'(\cos\theta)\sin\theta$$

$$E_\theta^{\text{out}} = -\frac{1}{r} \left.\frac{\partial\Phi_{\text{out}}}{\partial\theta}\right|_{r=a} = \frac{q}{4\pi\epsilon_0} \sum_l \left[\frac{a^{l-1}}{d^{l+1}} + \frac{\beta_l}{a} \right] P_l'(\cos\theta)\sin\theta$$

Matching these gives

$$\alpha_l = \frac{\epsilon}{\epsilon_0}\left[\frac{a^l}{d^{l+1}} + \beta_l\right] \tag{11}$$

On the other hand, for the perpendicular electric displacement, we have

$$\begin{aligned}
D_r^{\text{in}} &= -\epsilon\left.\frac{\partial\Phi_{\text{in}}}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\sum_l \frac{l\alpha_l}{a}P_l(\cos\theta)\\
D_r^{\text{out}} &= -\epsilon_0\left.\frac{\partial\Phi_{\text{out}}}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\sum_l \left[\frac{la^{l-1}}{d^{l+1}} - \frac{(l+1)\beta_l}{a}\right]P_l(\cos\theta)
\end{aligned}$$

Matching gives

$$\alpha_l = \frac{a^l}{d^{l+1}} - \frac{(l+1)}{l}\beta_l \tag{12}$$

Solving (11) and (12) yields

$$\begin{aligned}
\alpha_l &= \frac{2l+1}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{a^l}{d^{l+1}}\\
\beta_l &= \frac{(\frac{\epsilon_0}{\epsilon}-1)l}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{a^l}{d^{l+1}}
\end{aligned}$$

As a result, the interior and exterior potential, given by (9) and (10), has the form

$$\begin{aligned}
\Phi_{\text{in}} &= \frac{q}{4\pi\epsilon d}\sum_l \frac{2l+1}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\left(\frac{r}{d}\right)^l P_l(\cos\theta)\\
\Phi_{\text{out}} &= \frac{q}{4\pi\epsilon_0}\sum_l \left[\frac{r_<^l}{r_>^{l+1}} + \frac{(\frac{\epsilon_0}{\epsilon}-1)l}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)
\end{aligned} \tag{13}$$

Note that, without the dielectric sphere (so that $\epsilon = \epsilon_0$), this reduces to

$$\begin{aligned}
\Phi_{\text{in}} &= \frac{q}{4\pi\epsilon_0}\sum_l \frac{r^l}{d^{l+1}}P_l(\cos\theta)\\
\Phi_{\text{out}} &= \frac{q}{4\pi\epsilon_0}\sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)
\end{aligned}$$

which is simply the free space result

$$\Phi = \frac{q}{4\pi\epsilon_0|\vec{x} - d\hat{z}|}$$

b) Calculate the rectangular components of the electric field *near* the center of the sphere.

Near the center, we may expand $\Phi_{\rm in}$ in (13).

$$\begin{aligned}\Phi_{\rm in} &= \frac{q}{4\pi\epsilon d}\left[\frac{1}{\frac{\epsilon_0}{\epsilon}} + \frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\frac{r}{d}\cos\theta + \frac{5}{2+3\frac{\epsilon_0}{\epsilon}}\left(\frac{r}{d}\right)^2\frac{3\cos^2\theta-1}{2}+\cdots\right]\\ &= \frac{q}{4\pi\epsilon_0 d}\left[1+\frac{3}{2+\frac{\epsilon}{\epsilon_0}}\frac{z}{d}+\frac{5}{3+2\frac{\epsilon}{\epsilon_0}}\frac{3z^2-r^2}{2d^2}+\cdots\right]\end{aligned}$$

The electric field is then

$$\vec{E} = -\vec{\nabla}\Phi_{\rm in} = \frac{q}{4\pi\epsilon_0 d^2}\left[-\frac{3}{2+\frac{\epsilon}{\epsilon_0}}\hat{z}+\frac{5}{3+2\frac{\epsilon}{\epsilon_0}}\frac{x\hat{x}+y\hat{y}-2z\hat{z}}{d}+\cdots\right]$$

Very close to the center, the field is nearly uniform, and pointed in the $-\hat{z}$ direction (assuming $q>0$). The presence of the dielectric modifies the point charge result $\vec{E} = -(q/4\pi\epsilon_0 d^2)\hat{z}$ by the factor $3/(2+\epsilon/\epsilon_0)$.

c) Verify that, in the limit $\epsilon/\epsilon_0 \to \infty$, your result is the same as that for the conducting sphere.

For $\epsilon/\epsilon_0 \to \infty$, the potential (13) reduces to

$$\begin{aligned}\Phi_{\rm in} &= \frac{q}{4\pi\epsilon_0 d}\\ \Phi_{\rm out} &= \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} - \sum_{l=1}^{\infty}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)\\ &= \frac{q(a/d)}{4\pi\epsilon_0 r}+\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\frac{1}{r_>^{l+1}}P_l(\cos\theta)\\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{a/d}{r}+\frac{1}{|\vec{x}-d\hat{z}|}-\frac{a/d}{|\vec{x}-(a^2/d)\hat{z}|}\right]\end{aligned}$$

which is indeed the correct result for a conducting sphere. Note that the $l=0$ term in the sum had to be treated with care when taking the limit $\epsilon/\epsilon_0 \to \infty$. This results in the $q(a/d)/4\pi\epsilon_0 r$ term corresponding to an uncharged (and ungrounded) conducting sphere having non-zero potential when the charge q is brought near it.