Physics 505 **Fall 2007**

Homework Assignment #5 — Solutions

Textbook problems: Ch. 3: 3.13, 3.17, 3.26, 3.27

3.13 Solve for the potential in Problem 3.1, using the appropriate Green function obtained in the text, and verify that the answer obtained in this way agrees with the direct solution from the differential equation.

Recall that Problem 3.1 asks for the potential between two concentric spheres of radii a and b (with $b > a$), where the upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V, and where the other hemispheres are at zero potential. Since this problem involves the potential between two spheres, we use the Dirichlet Green's function

$$G(\vec{x}, \vec{x}\,') = \sum_{l,m} \frac{4\pi}{2l+1} \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) Y_l^m(\Omega) Y_l^{m\,*}(\Omega') \tag{1}$$

Because there are no charges between the spheres, the Green's function solution for the potential only involves the surface integral

$$\Phi(\vec{x}\,) = -\frac{1}{4\pi}\int_S \Phi(\vec{x}\,')\frac{\partial G}{\partial n'} da'$$

Here, we note a subtlety in that the boundary surface is actually disconnected, and includes both the inner sphere of radius a and the outer sphere of radius b. This means that the potential may be written as a sum of two contributions

$$\Phi(\vec{x}\,) = -\frac{1}{4\pi}\int_{r'=a} \Phi(\vec{x}\,')\frac{\partial G}{\partial n'} a^2 d\Omega' - \frac{1}{4\pi}\int_{r'=b} \Phi(\vec{x}\,')\frac{\partial G}{\partial n'} b^2 d\Omega' \tag{2}$$

We now compute the normal derivatives of the Green's function (1)

$$\begin{aligned}\left.\frac{\partial G}{\partial n'}\right|_{r'=a} &= -\left.\frac{\partial G}{\partial r'}\right|_{r_<=r'=a} = \\ &= -\frac{4\pi}{a^2}\sum_{l,m}\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right] Y_l^m(\Omega) Y_l^{m\,*}(\Omega')\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial G}{\partial n'}\right|_{r'=b} &= \left.\frac{\partial G}{\partial r'}\right|_{r_>=r'=b} = \\ &= -\frac{4\pi}{b^2}\sum_{l,m}\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right] Y_l^m(\Omega) Y_l^{m\,*}(\Omega')\end{aligned}$$

Inserting these expressions into (2) yields

$$\begin{aligned}
&\Phi(\vec{x}\,) \\
&= \sum_{l,m}\left[\int V_a(\Omega')Y_l^{m\,*}(\Omega')d\Omega'\right]\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right]Y_l^m(\Omega) \\
&\quad+\sum_{l,m}\left[\int V_b(\Omega')Y_l^{m\,*}(\Omega')d\Omega'\right]\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{r}{b}\right)^{l}-\left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right]Y_l^m(\Omega)
\end{aligned}$$

This is the general expression for the solution to the boundary value problem where $V_a(\Omega)$ is the potential on the inner sphere and $V_b(\Omega)$ is the potential on the outer sphere.

For the upper/lower hemispheres problem, we note that azimuthal symmetry allows us to restrict the m values to $m = 0$ only. In this case, the spherical harmonic expansion reduces to a Legendre polynomial expansion

$$\begin{aligned}
&\Phi(\vec{x}\,) \\
&= \sum_{l}\left[\int_{-1}^{1} V_a(\zeta)P_l(\zeta)d\zeta\right]\frac{2l+1}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right]P_l(\cos\theta) \\
&\quad+\sum_{l}\left[\int V_b(\zeta)P_l(\zeta)d\zeta\right]\frac{2l+1}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(\frac{r}{b}\right)^{l}-\left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta)
\end{aligned}$$

where $\zeta = \cos\theta'$. Since $V_a = V$ for $\zeta > 0$ and $V_b = V$ for $\zeta < 0$, this above expression reduces to

$$\begin{aligned}
\Phi(\vec{x}\,) &= \sum_{l}\frac{(2l+1)VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right]P_l(\cos\theta) \\
&\quad+\sum_{l}\frac{(2l+1)(-1)^lVN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(\frac{r}{b}\right)^{l}-\left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta) \\
&= \sum_{l}\frac{(2l+1)VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\Bigg[\left(\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right) \\
&\qquad\qquad+(-1)^l\left(\left(\frac{r}{b}\right)^{l}-\left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right)\Bigg]P_l(\cos\theta)
\end{aligned}$$

where

$$N_l = \int_0^1 P_l(\zeta)d\zeta = \begin{cases} 1 & l = 0 \\ (-1)^{j+1}\dfrac{\Gamma(j-\frac{1}{2})}{2\sqrt{\pi}j!} & l = 2j-1 \text{ odd} \end{cases}$$

If desired, the potential may be rearranged to read

$$\Phi(\vec{x}\,) = \sum_l \frac{(2l+1)VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(1+(-1)^{l+1}\left(\frac{a}{b}\right)^l\right)\left(\frac{a}{r}\right)^{l+1}\right.$$
$$\left. + (-1)^l\left(1+(-1)^{l+1}\left(\frac{a}{b}\right)^{l+1}\right)\left(\frac{r}{b}\right)^l\right]P_l(\cos\theta)$$
$$= \frac{V}{2} + V\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(4j-1)\Gamma(j-\frac{1}{2})}{4\sqrt{\pi}j!\left(1-\left(\frac{a}{b}\right)^{4j-1}\right)}\left[\left(1+\left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j}\right.$$
$$\left. - \left(1+\left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1}\right]P_{2j-1}(\cos\theta)$$

which agrees with the solution to Problem 3.1 that we have found earlier.

3.17 The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

a) Using cylindrical coordinates show that one form of the Green function is

$$G(\vec{x},\vec{x}\,')$$
$$= \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)$$

In cylindrical coordinates, the polar direction ϕ is periodic with period 2π. This suggests that the Green's function could be expanded as a Fourier series in $e^{im\phi}$. Similarly, the boundary conditions $G = 0$ at $z = 0$ and $z = L$ motivates the use of a Fourier sine series $\sin(n\pi z/L)$ in the z coordinate. More precisely, a complete Fourier expansion in ϕ and z would give

$$G(\vec{x},\vec{x}\,') = \sum_{m,n,m',n'} g(\rho,\rho')e^{im\phi}e^{im'\phi'}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n'\pi z}{L}\right)$$

However, it turns out that m and m' (and n and n') do not need to be chosen to be independent. This can be seen from the Green's function equation (given here as a differential equation in $\vec{x}\,$)

$$\nabla_x G(\vec{x},\vec{x}\,') = -4\pi\delta^3(\vec{x}-\vec{x}\,')$$

In cylindrical coordinates, this reads

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]G(\rho,\phi,z;\rho',\phi',z') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \quad (3)$$

Using the completeness relations

$$\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = 2\pi\delta(\phi-\phi') \tag{4}$$

and

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) = \frac{L}{2}\delta(z-z')$$

suggests that we take

$$G(\vec{x},\vec{x}\,') = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} g(\rho,\rho')e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \tag{5}$$

Substituting this decomposition into (3) gives

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \left(\frac{n\pi}{L}\right)^2\right] g(\rho,\rho') = -\frac{4}{L\rho}\delta(\rho,\rho')$$

Making the substitution

$$x = \frac{n\pi\rho}{L}$$

converts (the homogeneous part of) this to a modified Bessel equation

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(1+\frac{m^2}{x^2}\right)\right] g(x,x') = -\frac{4}{Lx}\delta(x,x')$$

At this stage, the solution becomes standard. Noting that the modified Bessel function $I_m(x)$ blows up as $x \to \infty$ and the function $K_m(x)$ blows up as $x \to 0$, we are left with

$$g(x,x') = \begin{cases} AI_m(x) & x < x' \\ BK_m(x) & x > x' \end{cases}$$

where the coefficients A and B are determined by the matching conditions

$$g_< = g_>, \qquad \frac{d}{dx}g_< = \frac{d}{dx}g_> + \frac{4}{Lx'}$$

at $x = x'$. This system may be solved to yield

$$A = \frac{4}{Lx'}\frac{K_m(x')}{I'_m(x')K_m(x') - I_m(x')K'_m(x')}$$

$$B = \frac{4}{Lx'}\frac{I_m(x')}{I'_m(x')K_m(x') - I_m(x')K'_m(x')}$$

Noting that the modified Bessel functions satisfy the Wronskian formula

$$I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -\frac{1}{x}$$

finally gives

$$\begin{aligned} g(x,x') &= \frac{4}{L}\begin{cases} I_m(x)K_m(x') & x < x' \\ I_m(x')K_m(x) & x > x' \end{cases} \\ &= \frac{4}{L} I_m(x_<)K_m(x_>) \end{aligned}$$

where

$$x_< = \min(x,x'), \qquad x_> = \max(x,x')$$

Converting x back to ρ and substituting into (5) then gives the desired Dirichlet Green's function

$$G(\vec{x},\vec{x}\,') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi\rho_<}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right)$$

b) Show that an alternative form of the Green function is

$$G(\vec{x},\vec{x}\,') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}$$

This alternative form of the Green's function is derived by expanding in ϕ and ρ instead of ϕ and z. For the ρ expansion, we use the integral relation

$$\int_0^\infty kJ_\nu(k\rho)J_\nu(k\rho')dk = \frac{1}{\rho}\delta(\rho-\rho')$$

along with the completeness relation (4) to motivate the decomposition

$$G(\vec{x},\vec{x}\,') = \sum_{m=-\infty}^{\infty}\int_0^\infty kdk\, g_k(z,z')e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho') \tag{6}$$

Since the Bessel function $J_m(k\rho)$ satisfies the Bessel equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)\right] J_m(k\rho) = 0$$

the substitution of (6) into the Greens' function equation (3) gives

$$\left[\frac{d^2}{dz^2} - k^2\right] g_k(z,z') = -2\delta(z-z')$$

Since $g_k(z, z')$ vanishes at $z = 0$ and $z = L$, this is a standard one-dimensional Green's function problem. Writing

$$g_k(z, z') = \begin{cases} A\sinh(kz) & z < z' \\ B\sinh[k(L-z)] & z > z' \end{cases}$$

we find that the matching and jump conditions become

$$A\sinh(kz') = B\sinh[k(L-z')], \qquad A\cosh(kz') = -B\cosh[k(L-z')] + \frac{2}{k}$$

This may be solved to give

$$A = \frac{2}{k}\frac{\sinh[k(L-z')]}{\sinh(kL)}, \qquad B = \frac{2}{k}\frac{\sinh(kz')}{\sinh(kL)}$$

so that

$$g_k(z, z') = \frac{2}{k\sinh(kL)}\sinh(kz_<)\sinh[k(L-z_>)]$$

Substituting this into (6) then yields

$$G(\vec{x}, \vec{x}\,) = 2\sum_{m=-\infty}^{\infty}\int_0^k dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}$$

3.26 Consider the Green function appropriate for Neumann boundary conditions for the volume V between the concentric spherical surfaces defined by $r = a$ and $r = b$, $a < b$. To be able to use (1.46) for the potential, impose the simple constraint (1.45). Use an expansion in spherical harmonics of the form

$$G(\vec{x}, \vec{x}\,') = \sum_{l=0}^{\infty} g_l(r, r')P_l(\cos\gamma)$$

where $g_l(r, r') = r_<^l/r_>^{l+1} + f_l(r, r')$.

a) Show that for $l > 0$, the radial Green function has the symmetric form

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{(b^{2l+1} - a^{2l+1})}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]$$

There are several approaches to this problem. However, we first consider the Neumann boundary condition (1.45)

$$\left.\frac{\partial G(\vec{x}\,\vec{x}\,')}{\partial n'}\right|_{\text{bndy}} = -\frac{4\pi}{S}$$

For this problem with two boundaries, the surface area S must be the area of both boundaries (ie it is the *total* area surrounding the volume). Hence $S = 4\pi(a^2+b^2)$, and in particular this is uniform (constant) in the angles. As a result, this will only contribute to the $l = 0$ term in the expansion of the Green's function. More precisely, we could write

$$\left.\frac{\partial G(\vec{x}\,\vec{x}\,')}{\partial n'}\right|_{\rm bndy} = \left.\sum_l \frac{\partial g_l(r,r')}{\partial n'} P_l(\cos\gamma)\right|_{\rm bndy} = -\frac{1}{a^2+b^2} P_0(\cos\gamma)$$

Since the Legendre polynomials are orthogonal, this implies that

$$\left.\frac{\partial g_l(r,r')}{\partial n'}\right|_{\rm bndy} = -\frac{1}{a^2+b^2}\delta_{l,0}$$

Noting that the outward normal is either in the $-\hat{r}'$ or the $\hat{r}'$ direction for the sphere at a or b, respectively, we end up with two boundary condition equations

$$\left.\frac{\partial g_l(r,r')}{\partial r'}\right|_a = \frac{1}{a^2+b^2}\delta_{l,0} \qquad \left.\frac{\partial g_l(r,r')}{\partial r'}\right|_b = -\frac{1}{a^2+b^2}\delta_{l,0} \tag{7}$$

Now that we have written down the boundary conditions for $g_l(r,r')$, we proceed to obtain its explicit form. The suggestion of the problem is to write

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + f_l(r,r')$$

Since

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

we see that the first term in $g_l(r,r')$ is designed to give the singular source delta function. The remaining term

$$F(\vec{x},\vec{x}\,') = \sum_l f_l(r,r') P_l(\cos\gamma)$$

then solves the homogeneous equation $\nabla^2_{x'} F(\vec{x},\vec{x}\,') = 0$. But we know how to solve Laplace's equation in spherical coordinates, and the result is that the radial function must be of the form

$$f_l(r,r') = A_l r'^l + B_l \frac{1}{r'^{l+1}}$$

Note that we are taking the Green's function equation to act on the $\vec{x}\,'$ variable, where $\vec{x}$ may be thought of as a parameter (constant) giving the location of the delta function source. We thus have

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + A_l r'^l + B_l \frac{1}{r'^{l+1}} \tag{8}$$

All that remains is to use the boundary conditions (7) to solve for A_l and B_l. For the inside sphere (at a), we have

$$l\frac{a^{l-1}}{r^{l+1}} + lA_l a^{l-1} - (l+1)B_l\frac{1}{a^{l+2}} = \frac{\delta_{l,0}}{a^2+b^2} \tag{9}$$

while for the outside sphere we have

$$-(l+1)\frac{r^l}{b^{l+2}} + lA_l b^{l-1} - (l+1)B_l\frac{1}{b^{l+2}} = -\frac{\delta_{l,0}}{a^2+b^2} \tag{10}$$

For $l \neq 0$ we rewrite these equations as

$$\begin{pmatrix} la^{2l+1} & -(l+1) \\ lb^{2l+1} & -(l+1) \end{pmatrix}\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} -la^{2l+1}/r^{l+1} \\ (l+1)r^l \end{pmatrix}$$

which may be solved to give

$$\begin{aligned}\begin{pmatrix} A_l \\ B_l \end{pmatrix} &= \frac{1}{l(l+1)(b^{2l+1}-a^{2l+1})}\begin{pmatrix} -(l+1) & (l+1) \\ -lb^{2l+1} & la^{2l+1} \end{pmatrix}\begin{pmatrix} -la^{2l+1}/r^{l+1} \\ (l+1)r^l \end{pmatrix} \\ &= \frac{r^l}{b^{2l+1}-a^{2l+1}}\begin{pmatrix} (a/r)^{2l+1} + (l+1)/l \\ a^{2l+1} + l/(l+1)(ab/r)^{2l+1} \end{pmatrix}\end{aligned}$$

Inserting this into (8) yields

$$\begin{aligned} g_l(r,r') &= \frac{r_<^l}{r_>^{l+1}} + \frac{r^l}{b^{2l+1}-a^{2l+1}}\Bigg[\left(\left(\frac{a}{r}\right)^{2l+1} + \frac{l+1}{l}\right)r'^l \\ &\qquad + \left(a^{2l+1} + \frac{l}{l+1}\left(\frac{ab}{r}\right)^{2l+1}\right)\frac{1}{r'^{l+1}}\Bigg] \\ &= \frac{r_<^l}{r_>^{l+1}} \\ &\quad + \frac{1}{b^{2l+1}-a^{2l+1}}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r'^l}{r^{l+1}} + \frac{r^l}{r'^{l+1}}\right)\right] \\ &= \frac{1}{b^{2l+1}-a^{2l+1}}\Bigg[\frac{l+1}{l}(r_<r_>)^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(r_<r_>)^{l+1}} \\ &\qquad + b^{2l+1}\frac{r_<^l}{r_>^{l+1}} + a^{2l+1}\frac{r_>^l}{r_<^{l+1}}\Bigg] \\ &= \frac{l+1}{l(b^{2l+1}-a^{2l+1})}\left(r_<^l + \frac{l}{l+1}\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(r_>^l + \frac{l}{l+1}\frac{b^{2l+1}}{r_>^{l+1}}\right) \end{aligned} \tag{11}$$

which is valid for $l \neq 0$. Note that in the last few lines we have been able to rewrite the Green's function in terms of a product of $u(r_<)$ and $v(r_>)$ where u and v satisfies Neumann boundary conditions at $r = a$ and $r = b$, respectively.

This is related to another possible method of solving this problem. Using the Legendre identity

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\cos\gamma) = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

the Green's function equation may be reduced to the one-dimensional problem

$$\left[\frac{d}{dr'} r'^2 \frac{d}{dr'} - l(l+1)\right] g_l(r, r') = -(2l+1)\delta(r - r')$$

Using the general method for the Sturm-Liouville problem, the Green's function is given by

$$g_l(r, r') = -\frac{2l+1}{A_l} u_l(r_<) v_l(r_>) \tag{12}$$

where $u(r')$ and $v(r')$ solve the homogeneous equation and the constant A_l is fixed by the Wronskian, $W(u, v) = A_l/r'^2$. For $l \neq 0$ the boundary conditions (7) are homogeneous

$$u'(r')\big|_{r'=a} = 0 \qquad v'(r')\big|_{r'=b} = 0$$

It is easy to see that these are satisfied by

$$u(r') = r'^l + \frac{l}{l+1} \frac{a^{2l+1}}{r'^{l+1}} \qquad v(r') = r'^l + \frac{l}{l+1} \frac{b^{2l+1}}{r'^{l+1}}$$

Computing the Wronskian gives

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \frac{l(2l+1)(a^{2l+1} - b^{2l+1})}{(l+1)r'^2}$$

which allows us to identify

$$A_l = -(2l+1)\frac{l}{l+1}(b^{2l+1} - a^{2l+1})$$

This gives the result of the last line of (11).

b) Show that for $l = 0$

$$g_0(r, r') = \frac{1}{r_>} - \left(\frac{a^2}{a^2 + b^2}\right)\frac{1}{r'} + f(r)$$

where $f(r)$ is arbitrary. Show explicitly in (1.46) that answers for the potential $\Phi(\vec{x}\,)$ are independent of $f(r)$.

The $l = 0$ case involves a non-homogeneous boundary condition. Hence the result of (12) will not work. Of course, we can still work out the one-dimensional delta function problem with matching and jump conditions at $r' = r$. However it is more direct to return to (9) and (10) and to simply solve those conditions for $l = 0$. Both (9) and (10) result in

$$B_0 = -\frac{a^2}{a^2 + b^2}$$

while leaving A_0 completely undetermined. Finally, since r is thought of as a parameter, this indicates that $A_0 = f(r)$ can be an arbitrary function of r. The $l = 0$ Green's function is given by (8)

$$g_0(r, r') = \frac{1}{r_>} - \frac{a^2}{a^2 + b^2}\frac{1}{r'} + f(r)$$

Incidentally, we note that without the inhomogeneous Neumann boundary condition term $-4\pi/S$ there will be no solution to the system (9) and (10) for $l = 0$ (unless b is taken to ∞). This demonstrates the inconsistency of simply setting $\partial G/\partial n' = 0$ for the Neumann Green's function.

Note that, by setting $f(r) = -a^2/[(a^2 + b^2)r]$ we obtain a symmetrical Green's function

$$g_0(r, r') = \frac{1}{r_>} - \frac{a^2}{a^2 + b^2}\left(\frac{1}{r'} + \frac{1}{r}\right)$$

On the other hand, the choice of $f(r)$ is unphysical. This arises because, for the Neumann Green's function, the $f(r)$ contribution to the potential is given by

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')f(r)\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}f(r)\,da' \\ &= \frac{f(r)}{4\pi\epsilon_0}\left(\int_V \rho(\vec{x}\,')\,d^3x' - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot d\hat{a}'\right) \\ &= \frac{f(r)}{4\pi\epsilon_0}\left(q_{\text{enc}} - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot d\hat{a}'\right) = 0\end{aligned}$$

by Gauss' law. It is important not to mix up r and r' in this derivation.

3.27 Apply the Neumann Green function of Problem 3.26 to the situation in which the normal electric field is $E_r = -E_0\cos\theta$ at the outer surface ($r = b$) and is $E_r = 0$ on the inner surface ($r = a$).

a) Show that the electrostatic potential inside the volume V is

$$\Phi(\vec{x}\,) = E_0\frac{r\cos\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right)$$

where $p = a/b$. Find the components of the electric field

$$E_r(r, \theta) = -E_0\frac{\cos\theta}{1 - p^3}\left(1 - \frac{a^3}{r^3}\right), \qquad E_\theta(r, \theta) = E_0\frac{\sin\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right)$$

Since there is no charge between the spheres, the solution to be boundary value problem is given by

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G(\vec{x},\vec{x}\,')\,da' \\
&= -\frac{1}{4\pi}\int_{r'=b} E_r(\Omega')G(\vec{x},\vec{x}\,')b^2\,d\Omega' \\
&= \frac{E_0 b^2}{4\pi}\int_{r'=b} G(\vec{x},\vec{x}\,')\cos\theta'\,d\Omega' \\
&= \frac{E_0 b^2}{4\pi}\sum_{l=0}^{\infty}\int_{r'=b} g_l(r,r')P_l(\cos\gamma)\cos\theta'\,d\Omega'
\end{aligned}$$

By writing

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_m Y_l^m(\Omega)Y_l^{m\,*}(\Omega')$$

and noting that $\cos\theta = \sqrt{4\pi/3}Y_1^0(\Omega)$, we end up with the expansion

$$\begin{aligned}
\Phi(\vec{x}\,) &= E_0 b^2\sqrt{\frac{4\pi}{3}}\sum_{l,m}\frac{g_l(r,b)Y_l^m(\Omega)}{2l+1}\int Y_l^{m\,*}(\Omega')Y_1^0(\Omega')d\Omega' \\
&= E_0 b^2\sqrt{\frac{4\pi}{3}}\frac{g_1(r,b)Y_1^0(\Omega)}{3} \\
&= \frac{E_0 b^2\cos\theta}{3}g_1(r,b)
\end{aligned}$$

where we have used orthogonality of the spherical harmonics. Inserting $l = 1$ into (11) then gives

$$\Phi(\vec{x}\,) = \frac{E_0 b^2\cos\theta}{3}\frac{2}{b^3-a^3}\left(r+\frac{a^3}{2r^2}\right)\frac{3b}{2} = \frac{E_0 r\cos\theta}{1-(a/b)^3}\left(1+\frac{a^3}{2r^3}\right) \qquad (13)$$

This is the potential for a constant electric field combined with an electric dipole. Defining $p = a/b$, the components of the electric field are

$$E_r = -\frac{\partial\Phi}{\partial r} = -\frac{E_0\cos\theta}{1-p^3}\left(1-\frac{a^3}{r^3}\right), \qquad E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{E_0\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

Note that the boundary conditions $E_r\big|_{r=a} = 0$ and $E_r\big|_{r=b} = -E_0\cos\theta$ are obviously satisfied. On the other hand, the parallel component of the field, E_θ, is non-vanishing on both surfaces (except at the poles). Physically, this indicates that these surfaces are not conductors.

b) Calculate the Cartesian or cylindrical components of the field, E_z ad E_ρ, and make a sketch or computer plot of the lines of electric force for a typical case of $p = 0.5$.

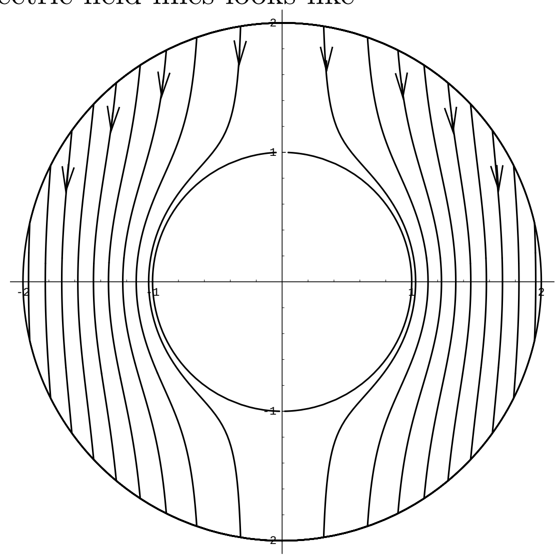
Rewriting (13) as

$$\Phi(\vec{x}\,) = \frac{E_0}{1-p^3}\left(z + \frac{a^3 z}{2r^3}\right) = \frac{E_0}{1-p^3}\left(z + \frac{a^3 z}{2(\rho^2+z^2)^{3/2}}\right)$$

we obtain

$$E_z = -\frac{\partial\Phi}{\partial z} = -\frac{E_0}{1-p^3}\left(1 + \frac{a^3(1-3\hat{z}^2)}{2r^3}\right)$$
$$E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\frac{E_0}{1-p^3}\left(-\frac{3a^3\hat{z}\hat{\rho}}{2r^3}\right)$$

where $\hat{\rho} = \rho/r$, $\hat{z} = z/r$, and $r = \sqrt{\rho^2+z^2}$. As indicated above, this corresponds to a constant electric field combined with an electric dipole. For $E_0 > 0$, a sketch of the electric field lines looks like



This sketch indicates that the radial component of the electric field vanishes at the surface of the inner sphere.