Physics 505 **Fall 2007**

Homework Assignment #4 — Solutions

Textbook problems: Ch. 3: 3.1, 3.2, 3.4, 3.7

3.1 Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.

Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

The general expansion in Legendre polynomials is of the form

$$\Phi(r,\theta) = \sum_l [A_l r^l + B_l r^{-l-1}] P_l(\cos\theta) \tag{1}$$

Since we are working in the region between spheres, neither A_l nor B_l can be assumed to vanish. To solve for both A_l and B_l we will need to consider boundary conditions at $r = a$ and $r = b$

$$\Phi(a,\theta) = \sum_l [A_l a^l + B_l a^{-l-1}] P_l(\cos\theta) = \begin{cases} V & \cos\theta \ge 0 \\ 0 & \cos\theta < 0 \end{cases}$$

$$\Phi(b,\theta) = \sum_l [A_l b^l + B_l b^{-l-1}] P_l(\cos\theta) = \begin{cases} 0 & \cos\theta > 0 \\ V & \cos\theta \le 0 \end{cases}$$

Using orthogonality of the Legendre polynomials, we may write

$$\begin{aligned} A_l a^l + B_l a^{-l-1} &= \frac{2l+1}{2} V \int_0^1 P_l(x)\,dx \\ A_l b^l + B_l b^{-l-1} &= \frac{2l+1}{2} V \int_{-1}^0 P_l(x)\,dx = \frac{2l+1}{2} V (-1)^l \int_0^1 P_l(x)\,dx \end{aligned}$$

where in the last expression we used the fact that $P_l(-x) = (-1)^l P_l(x)$. Since the integral is only over half of the standard interval, it does not yield a particularly simple result. For now, we define

$$N_l = \int_0^1 P_l(x)\,dx \tag{2}$$

As a result, we have the system of equations

$$\begin{pmatrix} a^l & a^{-l-1} \\ b^l & b^{-l-1} \end{pmatrix} \begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{2l+1}{2} V N_l \begin{pmatrix} 1 \\ (-1)^l \end{pmatrix}$$

which may be solved to give

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{2l+1}{2} V N_l \frac{1}{b^{2l+1} - a^{2l+1}} \begin{pmatrix} (-1)^l b^{l+1} - a^{l+1} \\ (ab)^{l+1}(b^l + (-1)^{l+1} a^l) \end{pmatrix}$$

Inserting this into (1) gives

$$\begin{aligned} \Phi(r,\theta) = \tfrac{1}{2} V \sum_l \frac{(2l+1)N_l}{1-\left(\frac{a}{b}\right)^{2l+1}} \Bigg[& (-1)^l \left(1 + (-1)^{l+1} \left(\frac{a}{b}\right)^{l+1}\right) \left(\frac{r}{b}\right)^l \\ & + \left(1 + (-1)^{l+1}\left(\frac{a}{b}\right)^l\right)\left(\frac{a}{r}\right)^{l+1}\Bigg] P_l(\cos\theta) \end{aligned} \tag{3}$$

We now examine the integral (2). First note that for even l we may actually extend the region of integration

$$N_{2j} = \int_0^1 P_{2j}(x)\,dx = \tfrac{1}{2}\int_{-1}^1 P_{2j}(x)\,dx = \tfrac{1}{2}\int_{-1}^1 P_0(x)P_{2j}(x)\,dx = \delta_{j,0}$$

This demonstrates that the only contribution from even l is for $l = 0$, corresponding to the average potential. Using this fact, the potential (3) reduces to

$$\begin{aligned} \Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{j=1}^{\infty} \frac{(4j-1)N_{2j-1}}{1-\left(\frac{a}{b}\right)^{4j-1}} \Bigg[& -\left(1 + \left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \\ & + \left(1 + \left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j}\Bigg] P_{2j-1}(\cos\theta) \end{aligned}$$

Physically, once the average $V/2$ is removed, the remaining potential is odd under the flip $z \to -z$ or $\cos\theta \to -\cos\theta$. This is why only odd Legendre polynomials may contribute.

Note that an alternative method of solution would be to use linear superposition

$$\Phi = \Phi_{\text{inner}} + \Phi_{\text{outer}}$$

where Φ_{inner} is the solution where the inner sphere has potential $V_a(\theta)$ and the outer sphere is grounded, and where Φ_{outer} is the solution where the outer sphere has potential $V_b(\theta)$ and the inner sphere is grounded. To calculate Φ_{inner} we note that the boundary conditions are such that $\Phi_{\text{inner}}(r = b) = 0$. This motivates an expansion of the form

$$\Phi_{\text{inner}}(r,\theta) = \sum_l \alpha_l \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) P_l(\cos\theta)$$

The boundary condition at $r = a$ is then

$$V_a(\theta) = \sum_l \frac{\alpha_l}{a^{l+1}}(1 - (a/b)^{2l+1})P_l(\cos\theta)$$

which, by orthogonality, gives

$$\alpha_l = \frac{2l+1}{2}\frac{a^{l+1}}{1-\left(\frac{a}{b}\right)^{2l+1}}\int_{-1}^{1} V_a(\cos\theta)P_l(\cos\theta)d(\cos\theta)$$

Similarly, for Φ_{outer}, we may interchange $a \leftrightarrow b$ and rearrange the expressions to obtain

$$\Phi_{\text{outer}}(r,\theta) = \sum_l \beta_l \left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$$

where

$$\beta_l = \frac{2l+1}{2}\frac{1/b^l}{1-\left(\frac{a}{b}\right)^{2l+1}}\int_{-1}^{1} V_b(\cos\theta)P_l(\cos\theta)d(\cos\theta)$$

Using $V_a = V$ for $\cos\theta > 0$ and $V_b = V$ for $\cos\theta < 0$ gives explicitly

$$\Phi_{\text{inner}} = \sum_l \frac{2l+1}{2}\frac{VN_l}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right] P_l(\cos\theta)$$

$$\Phi_{\text{outer}} = \sum_l \frac{2l+1}{2}\frac{(-1)^l VN_l}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right] P_l(\cos\theta)$$

When superposed, the solution is identical to (3) which we found above.

At this stage, we may simply perform elementary integrations to obtain the first few terms N_1, N_3, etc. However, we may derive a fairly simple expression for N_l by integrating the generating function

$$(1 - 2xt + t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)t^l$$

from $x = 0$ to 1. In other words

$$\sum_{l=0}^{\infty} N_l t^l = \int_0^1 (1 - 2xt + t^2)^{-1/2}dx = t^{-1}(-1 + t + \sqrt{1+t^2})$$

The square root yields a binomial expansion

$$(1+t^2)^{1/2} = 1 + \tfrac{1}{2}t^2 + \tfrac{1}{2}(-\tfrac{1}{2})\tfrac{1}{2!}t^4 + \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})\tfrac{1}{3!}t^6 + \cdots = 1 + \sum_{j=1}^{\infty}(-)^j \frac{\Gamma(j-\frac{1}{2})}{\Gamma(-\frac{1}{2})j!}t^{2j}$$

As a result

$$\sum_{l=0}^{\infty} N_l t^l = 1 + \sum_{j=1}^{\infty} (-)^{j+1} \frac{\Gamma(j-\frac{1}{2})}{2\sqrt{\pi} j!} t^{2j-1}$$

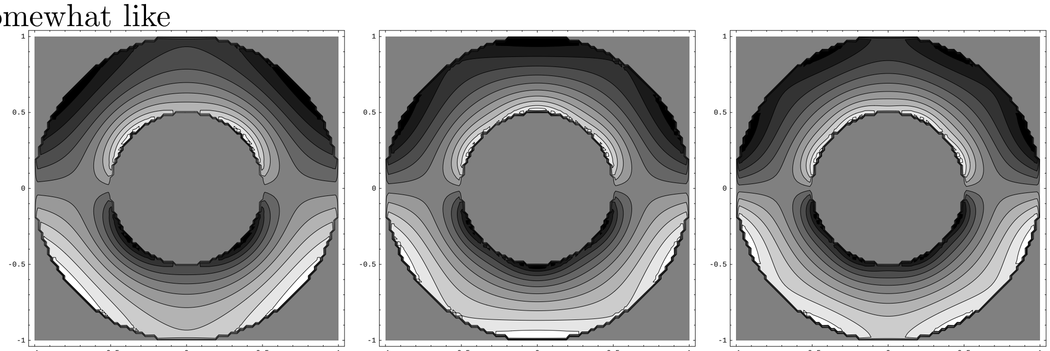
where we used the fact that $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$. Matching powers of t demonstrates that all even N_l terms vanish except $N_0 = 1$ and that

$$N_{2j-1} = (-1)^{j+1} \frac{\Gamma(j-\frac{1}{2})}{2\sqrt{\pi} j!}$$

The final result for the potential is thus

$$\begin{aligned}
\Phi(r,\theta) = \frac{V}{2} + V \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(4j-1)\Gamma(j-\frac{1}{2})}{4\sqrt{\pi} j! \left(1-\left(\frac{a}{b}\right)^{4j-1}\right)} \Bigg[&- \left(1+\left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \\
&+ \left(1+\left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j}\Bigg] P_{2j-1}(\cos\theta) \\
= \frac{V}{2} \\
+ V\Bigg[\frac{3}{4}\left(1-\left(\frac{a}{b}\right)^{3}\right)^{-1}&\left(-\left(1+\left(\frac{a}{b}\right)^{2}\right)\left(\frac{r}{b}\right)+\left(1+\left(\frac{a}{b}\right)\right)\left(\frac{a}{r}\right)^{2}\right)P_1(\cos\theta) \\
- \frac{7}{16}\left(1-\left(\frac{a}{b}\right)^{7}\right)^{-1}&\left(-\left(1+\left(\frac{a}{b}\right)^{4}\right)\left(\frac{r}{b}\right)^{3}+\left(1+\left(\frac{a}{b}\right)^{3}\right)\left(\frac{a}{r}\right)^{4}\right)P_3(\cos\theta) \\
+ \cdots \Bigg]
\end{aligned}$$

Taking a constant ϕ slice of the region between the spheres, the potential looks somewhat like



up to P_3 up to P_5 up to P_7

We note that including the higher Legendre modes improves the potential near the surfaces of the spheres. This is very much like summing the first few terms of a Fourier series. On the other hand, the potential midway between the spheres is well estimated by just the first term or two in the series. This is because both r/b and a/r are small in this region, and the series rapidly converges (assuming $a \ll b$, that is).

In the limit when $b \to \infty$ we may remove (a/b) and (r/b) terms. Removing the latter corresponds to having only inverse powers of r surviving, which is the expected case for an exterior solution. The result is

$$\Phi(r,\theta) \to \frac{V}{2} + \frac{V}{2}\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \cdots\right]$$

which agrees with the exterior solution for a sphere with oppositely charged hemispheres (except that here we have the average potential $V/2$ and that the potential difference between northern and southern hemispheres is only half as large).

Similarly, when $a \to 0$ we remove (a/b). But this time we get rid of the inverse powers (a/r) instead. The result is the interior solution

$$\Phi(r,\theta) \to \frac{V}{2} - \frac{V}{2}\left[\frac{3}{2}\left(\frac{r}{b}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \cdots\right]$$

which is again a reasonable result (this time with the hemispheres oppositely charged from the previous case).

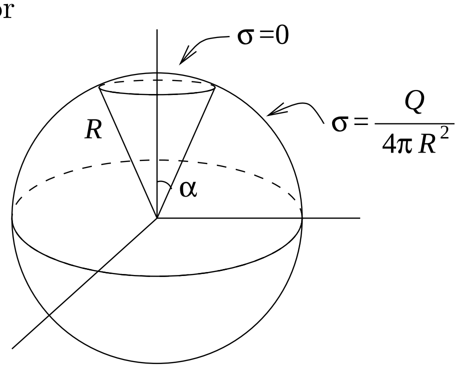
3.2 A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

Note that this problem specifies a spherical surface of charge, *not* a spherical conductor



We are thus interested in obtaining the potential $\Phi(r,\theta)$ given a charge distribution. This may be done using Coulomb's law (or, equivalently, integrating the Green's function with the charge density). Alternatively, in this problem, the surface charge density specifies an appropriate jump condition on the normal component of the electric field

$$E_{r\,\mathrm{out}}\Big|_{r=R} = E_{r\,\mathrm{in}}\Big|_{r=R} + \frac{1}{\epsilon_0}\sigma \tag{4}$$

This condition allows us to solve for the electrostatic potential $\Phi(r,\theta)$.

In particular, because of the azimuthal symmetry of this problem, we may perform an expansion in Legendre polynomials

$$\begin{aligned}\Phi_{\mathrm{in}} &= \sum_{l=0}^{\infty} \alpha_l \left(\frac{r}{R}\right)^l P_l(\cos\theta)\\ \Phi_{\mathrm{out}} &= \sum_{l=0}^{\infty} \alpha_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\end{aligned} \tag{5}$$

Note that the expansion coefficients α_l are identical for the inside and outside expansion. This holds because we demand that Φ is continuous at $r = R$. In this case, the radial components of the interior and exterior electric fields are given by

$$E_r = -\frac{\partial}{\partial r}\Phi \qquad \Rightarrow \qquad \begin{cases} E_{r\,\mathrm{in}} = -\displaystyle\sum_{l=1}^{\infty} \frac{l\alpha_l}{R}\left(\frac{r}{R}\right)^{l-1} P_l(\cos\theta)\\ E_{r\,\mathrm{out}} = \displaystyle\sum_{l=0}^{\infty} \frac{(l+1)\alpha_l}{R}\left(\frac{R}{r}\right)^{l+2} P_l(\cos\theta)\end{cases} \tag{6}$$

Substituting this into (4) gives

$$\sigma(\cos\theta) = \epsilon_0\big[E_{r\,\mathrm{out}} - E_{r\,\mathrm{in}}\big]_{r=R} = \sum_{l=0}^{\infty} \frac{(2l+1)\epsilon_0\alpha_l}{R} P_l(\cos\theta)$$

Since this is a Legendre polynomial expansion, the coefficients of the expansion are given by the relation

$$\frac{(2l+1)\epsilon_0\alpha_l}{R} = \frac{2l+1}{2}\int_{-1}^{1} \sigma(\cos\theta)P_l(\cos\theta)d(\cos\theta)$$

or

$$\alpha_l = \frac{R}{2\epsilon_0}\int_{-1}^{1} \sigma(\cos\theta)P_l(\cos\theta)d(\cos\theta)$$

Using

$$\sigma(\cos\theta) = \frac{Q}{4\pi R^2} \times \begin{cases} 0 & \cos\theta > \cos\alpha\\ 1 & \cos\theta < \cos\alpha\end{cases}$$

gives

$$\alpha_l = \frac{Q}{8\pi\epsilon_0 R}\int_{-1}^{\cos\alpha} P_l(\cos\theta)d(\cos\theta)$$

This may be integrated by using the Legendre polynomial relation

$$P_l(x) = \frac{1}{2l+1}(P'_{l+1}(x) - P'_{l-1}(x)) \tag{7}$$

where $P_{-1}(x)$ is formally defined to be a constant, so that $P'_{-1}(x) = 0$. In this case, we obtain

$$\alpha_l = \frac{Q}{8\pi\epsilon_0 R}\frac{1}{2l+1}\left[P_{l+1}(\cos\theta) - P_{l-1}(\cos\theta)\right]_{-1}^{\cos\alpha}$$

Noting that $P_l(-1) = (-1)^l$ then yields

$$\alpha_l = \frac{Q}{8\pi\epsilon_0 R}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]$$

so long as we define $P_{-1}(x) = -1$ (so that $P_1(-1) = P_{-1}(-1)$ is true). Substituting this into (5) then gives the desired potential (both inside and outside the spherical shell).

Note that, by defining

$$r_< = \min(r, R), \qquad r_> = \max(r, R)$$

the inside and outside expressions (5) may be combined into a compact form

$$\begin{aligned}\Phi &= \sum_{l=0}^{\infty}\alpha_l R\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\\ &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\end{aligned} \tag{8}$$

valid both inside and outside the shell.

b) Find the magnitude and the direction of the electric field at the origin.

By symmetry, the electric field at the origin must point along the $\hat{z}$ axis (either $\theta = 0$ or $\theta = \pi$). As a result, the radial component E_r given by (6) completely specifies the electric field at the origin. Noting that $E_{r\,\text{in}} \sim r^{l-1}$, we see that only the $l = 1$ component survives at the origin. As a result

$$\begin{aligned}E_r(r=0,\theta=0) &= -\frac{\alpha_1}{R}P_1(1)\\ &= -\frac{Q}{8\pi\epsilon_0 R^2}\frac{1}{3}[P_2(\cos\alpha) - P_0(\cos\alpha)]\\ &= -\frac{Q}{16\pi\epsilon_0 R^2}(\cos^2\alpha - 1) = \frac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\end{aligned}$$

In rectangular coordinates, this is equivalent to

$$\vec{E} = \frac{Q \sin^2 \alpha}{16\pi\epsilon_0 R^2} \hat{z} \tag{9}$$

Note that, had we chosen to look along the $-\hat{z}$ axis ($\theta = \pi$), we would have gotten an identical result since $P_1(\cos \pi) = -1$ would give an extra minus sign to compensate for the $-\hat{z}$ direction.

c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

We first consider the case $\alpha \to 0$, when the spherical cap becomes very small. For small α, we use $\cos \alpha \approx 1 - \frac{1}{2}\alpha^2$ as well as the Taylor expansion

$$P_l(\cos \alpha) \approx P_l(1 - \tfrac{1}{2}\alpha^2) \approx P_l(1) - \tfrac{1}{2}\alpha^2 P_l'(1) = 1 - 2\delta_{l,-1} - \tfrac{1}{2}\alpha^2 P_l'(1)$$

to write

$$P_{l+1}(\cos \alpha) - P_{l-1}(\cos \alpha) \approx 2\delta_{l,0} - \tfrac{1}{2}\alpha^2 [P_{l+1}'(1) - P_{l-1}'(1)]$$

Note that the delta functions take care of the special case concerning $P_{-1}(1) = -1$ instead of the usual $+1$. Using (7) now gives

$$P_{l+1}(\cos \alpha) - P_{l-1}(\cos \alpha) \approx 2\delta_{l,0} - \frac{2l+1}{2}\alpha^2 P_l(1) = 2\delta_{l,0} - \frac{2l+1}{2}\alpha^2$$

Substituting this into (8) yields

$$\Phi \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{r_>} - \frac{Q\alpha^2}{16\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos \theta)$$

Recalling the Green's function expansion

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos \gamma)$$

where $\cos \gamma = \hat{r} \cdot \hat{r}'$ finally gives

$$\Phi \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{r_>} - \frac{Q\alpha^2/4}{4\pi\epsilon_0} \frac{1}{|\vec{r} - R\hat{z}|}$$

Physically, this expression corresponds to the limit where the spherical shell is almost complete ($\Phi = Q/4\pi\epsilon_0 r_>$ for a shell centered at the origin). By linear superposition, the very small cap can be thought of effectively as an oppositely charged particle located at $R\hat{z}$ with charge given by

$$q = -\sigma dA = -\frac{Q}{4\pi R^2}(R^2 d\Omega) = -\frac{Q}{4\pi}(\pi\alpha^2) = -\frac{Q\alpha^2}{4}$$

The electric field at the origin is given by expanding (9) for $\alpha \approx 0$

$$\vec{E}(0) \approx \frac{Q\alpha^2/4}{4\pi\epsilon_0}\frac{\hat{z}}{R^2}$$

Again, this makes sense for the electric field of a particle of charge $-Q\alpha^2/4$ located at $R\hat{z}$. Note that the full spherical shell does not contribute any electric field, since we are inside the shell.

Finally, we consider the case $\alpha \to \pi$, when the spherical cap becomes very large. In this case, let $\alpha = \pi - \beta$ where β is the angle of the south polar cap. The Legendre polynomial expansion is now

$$P_l(\cos\alpha) = P_l(\cos(\pi - \beta)) = P_l(-\cos\beta) \approx P_l(-1 + \tfrac{1}{2}\beta^2) \approx (-1)^l + \tfrac{1}{2}\beta^2 P_l'(-1)$$

Note that the $l = -1$ special case is covered without any additions to this expression. This gives us

$$\begin{aligned} P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) &\approx \tfrac{1}{2}\beta^2[P_{l+1}'(-1) - P_{l-1}'(-1)] \\ &= \frac{2l+1}{2}\beta^2 P_l(-1) = \frac{2l+1}{2}\beta^2(-1)^l \end{aligned}$$

Substituting this into (8) gives

$$\begin{aligned} \Phi &\approx \frac{Q\beta^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}(-1)^l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) = \frac{Q\beta^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(-\cos\theta) \\ &= \frac{Q\beta^2/4}{4\pi\epsilon_0}\frac{1}{|\vec{r} + R\hat{z}|} \end{aligned}$$

This is clearly the potential due to a point charge of strength $Q\beta^2/4$ at the south pole $(-R\hat{z})$ of the spherical surface. For the electric field, we substitute $\alpha = \pi - \beta$ into (9) to obtain

$$\vec{E}(0) \approx \frac{Q\beta^2/4}{4\pi\epsilon_0}\hat{z}R^2$$

This is the electric field of a particle of charge $+Q\beta^2/4$ located at $-R\hat{z}$.

3.4 The surface of a hollow conducting sphere of inner radius a is divided into an *even number* of equal segments by a set of planes; their common line of intersection is the z axis and they are distributed uniformly in the angle ϕ. (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.

a) Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms, exhibit the coefficients as an integral over $\cos\theta$.

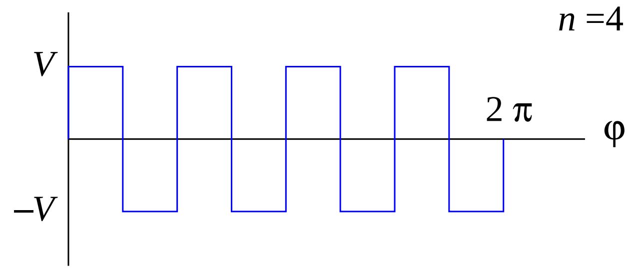
The general spherical harmonic expansion for the potential inside a sphere of radius a is

$$\Phi(r,\theta,\phi)=\sum_{l,m}\alpha_{lm}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)$$

where

$$\alpha_{lm}=\int V(\theta,\phi)Y^*_{lm}(\theta,\phi)d\Omega$$

In this problem, $V(\theta,\phi)=\pm V$ is independent of θ, but depends on the azimuthal angle ϕ. It can in fact be thought of as a square wave in ϕ



This has a familiar Fourier expansion

$$V(\phi)=\frac{4V}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin[(2k+1)n\phi]$$

This is already enough to demonstrate that the m values in the spherical harmonic expansion can only take on the values $\pm(2k+1)n$. In terms of associated Legendre polynomials, the expansion coefficients are

$$\begin{aligned}\alpha_{lm}&=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^{2\pi}V(\phi)e^{-im\phi}\,d\phi\int_{-1}^{1}P_l^m(x)\,dx\\&=\frac{4V}{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^{2\pi}\sin[(2k+1)n\phi]e^{-im\phi}\,d\phi\\&\qquad\qquad\times\int_{-1}^{1}P_l^m(x)\,dx\\&=-4iV\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\sum_{k=0}^{\infty}\frac{\delta_{m,(2k+1)n}-\delta_{m,-(2k+1)n}}{2k+1}\int_{-1}^{1}P_l^m(x)\,dx\end{aligned}$$

Using $P_l^{-m}(x) = (-)^m[(l-m)!/(l+m)!]P_l^m(x)$, we may write the non-vanishing coefficients as

$$\begin{aligned}\alpha_{l,-(2k+1)n} &= (-)^{n+1}\alpha_{l,(2k+1)n} \\ &= -\frac{4iV}{2k+1}\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1)n)!}{(l+(2k+1)n)!}}\int_{-1}^{1} P_l^{(2k+1)n}(x)\,dx\end{aligned} \tag{10}$$

for $k = 0, 1, 2, \ldots$. Since $l \geq (2k+1)n$, we see that the first non-vanishing term enters at order $l = n$. Making note of the parity of associated Legendre polynomials, $P_l^m(-x) = (-)^{l+m}P_l^m(x)$, we see that the non-vanishing coefficients are given by the sequence

$$\begin{array}{llll} \alpha_{n,n}, & \alpha_{n+2,n}, & \alpha_{n+4,n}, & \alpha_{n+6,n}, \ldots \\ \alpha_{3n,3n}, & \alpha_{3n+2,3n}, & \alpha_{3n+4,3n}, & \alpha_{3n+6,3n}, \ldots \\ \alpha_{5n,5n}, & \alpha_{5n+2,5n}, & \alpha_{5n+4,5n}, & \alpha_{5n+6,5n}, \ldots \\ \vdots \end{array}$$

b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

For $n = 1$, explicit computation shows that

$$\int_{-1}^{1} P_1^1(x)\,dx = -\frac{\pi}{2}, \qquad \int_{-1}^{1} P_3^1(x)\,dx = -\frac{3\pi}{16}, \qquad \int_{-1}^{1} P_3^3(x)\,dx = -\frac{45\pi}{8}$$

Inserting this into (10) yields

$$\begin{aligned}&\alpha_{1,-1} = \alpha_{1,1} = iV\sqrt{\frac{3\pi}{2}} \\ &\alpha_{3,-1} = \alpha_{3,1} = iV\sqrt{\frac{21\pi}{256}}, \qquad \alpha_{3,-3} = \alpha_{3,3} = iV\sqrt{\frac{35\pi}{256}}\end{aligned}$$

Hence

$$
\begin{aligned}
\Phi &= iV\left[\left(\frac{r}{a}\right)\sqrt{\frac{3\pi}{2}}(Y_{1,1}+Y_{1,-1})\right.\\
&\qquad\left.+\left(\frac{r}{a}\right)^3\left(\sqrt{\frac{21\pi}{256}}(Y_{3,1}+Y_{3,-1})+\sqrt{\frac{35\pi}{256}}(Y_{3,3}+Y_{3,-3})\right)+\cdots\right]\\
&= -2V\Im\left[\left(\frac{r}{a}\right)\sqrt{\frac{3\pi}{2}}Y_{1,1}+\left(\frac{r}{a}\right)^3\left(\sqrt{\frac{21\pi}{256}}Y_{3,1}+\sqrt{\frac{35\pi}{256}}Y_{3,3}\right)+\cdots\right]\\
&= 2V\Im\left[\left(\frac{r}{a}\right)\frac{3}{4}\sin\theta\, e^{i\phi}\right.\\
&\qquad\left.+\left(\frac{r}{a}\right)^3\left(\frac{21}{128}\sin\theta(5\cos^2\theta-1)e^{i\phi}+\frac{35}{128}\sin^3\theta\, e^{3i\phi}\right)+\cdots\right]\\
&= V\left[\left(\frac{r}{a}\right)\frac{3}{2}\sin\theta\sin\phi\right.\\
&\qquad\left.\left(\frac{r}{a}\right)^3\frac{7}{64}\big(3\sin\theta(5\cos^2\theta-1)\sin\phi+5\sin^3\theta\sin 3\phi\big)+\cdots\right]
\end{aligned}
\tag{11}
$$

To relate this to the previous result, we note that the way we have set up the wedges corresponds to taking the 'top' of the $+V$ hemisphere to point along the $\hat{y}$ axis. This may be rotated to the $\hat{z}'$ axis by a 90° rotation along the $\hat{x}$ axis. Explicitly, we take

$$\hat{y}=\hat{z}',\qquad \hat{z}=-\hat{y}',\qquad \hat{x}=\hat{x}'$$

or

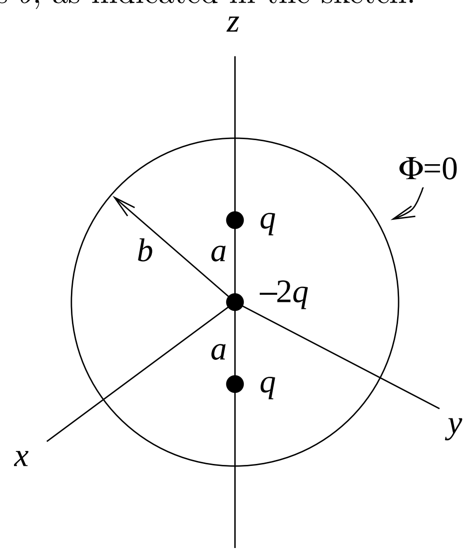
$$\sin\theta\sin\phi=\cos\theta',\qquad \cos\theta=-\sin\theta'\sin\phi',\qquad \sin\theta\cos\phi=\sin\theta'\cos\phi'$$

Noting that $\sin 3\phi=-\sin^3\phi+3\sin\phi\cos^2\phi$, the last line of (11) transforms into

$$
\begin{aligned}
\Phi &= V\left[\left(\frac{r}{a}\right)\frac{3}{2}\cos\theta'+\left(\frac{r}{a}\right)^3\frac{7}{64}\big(3\cos\theta'(5\sin^2\theta'\sin^2\phi'-1)\right.\\
&\qquad\left.+5(-\cos^3\theta'+3\cos\theta'\sin^2\theta'\cos^2\phi')\big)+\cdots\right]\\
&= V\left[\frac{3}{2}\left(\frac{r}{a}\right)\cos\theta'-\frac{7}{8}\left(\frac{r}{a}\right)^3\tfrac{1}{2}(5\cos^3\theta'-3\cos\theta')+\cdots\right]\\
&= V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta')-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3(\cos\theta')+\cdots\right]
\end{aligned}
$$

which reproduces the result (3.36).

3.7 Three point charges $(q, -2q, q)$ are located in a straight line with separation a and with the middle charge $(-2q)$ at the origin of a grounded conducting spherical shell of radius b, as indicated in the sketch.



a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

The potential for the above three point charges is given simply by

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r} - a\hat{z}|} + \frac{1}{|\vec{r} + a\hat{z}|}\right] \tag{12}$$

This may be expanded in Legendre polynomials using

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

where $\cos\gamma = \hat{r} \cdot \hat{r}'$ and where $r_<$ ($r_>$) is the smaller (greater) of r and r' to obtain

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} [P_l(\cos\theta) + P_l(-\cos\theta)]\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} [1 + (-1)^l] P_l(\cos\theta)\right] \\ &= \frac{q}{2\pi\epsilon_0}\left[-\frac{1}{r} + \sum_{l \text{ even}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right]\end{aligned}$$

Here $r_< r$ and $r_>$ are given by

$$r_< = \min(r, a), \qquad r_> = \max(r, a)$$

In order to take the limit $a \to 0$, we first set $r_< = a$ and $r_> = r$. This gives

$$\Phi(r > a) = \frac{q}{2\pi\epsilon_0}\left[-\frac{1}{r} + \sum_{l \text{ even}} \frac{a^l}{r^{l+1}} P_l(\cos\theta)\right] = \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

As $a \to 0$, the $l = 2$ term in the sum dominates over the others. Defining $qa^2 = Q$, we see that

$$\Phi \to \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos\theta) = \frac{Q}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$

This is an electrostatic quadrupole.

b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r, \theta, \phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)$$

The problem with a grounded sphere of radius b can be solved by the method of images. In particular, the image charge solution modifies (12) to

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r} - a\hat{z}|} + \frac{1}{|\vec{r} + a\hat{z}|} + \frac{2}{b} - \frac{b/a}{|\vec{r} - (b^2/a)\hat{z}|} - \frac{b/a}{|\vec{r} + (b^2/a)\hat{z}|}\right]$$

The Legendre polynomial expansion gives

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\left[\frac{2}{b} - \frac{2}{r} + \sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{b}{a}\frac{r^l}{(b^2/a)^{l+1}}\right)[P_l(\cos\theta) + P_l(-\cos\theta)]\right] \\ &= \frac{q}{2\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{r} + \sum_{l \text{ even}}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta)\right]\end{aligned}$$

For $r > a$, this reads

$$\begin{aligned}\Phi(r > a) &= \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots}\left(\frac{a^l}{r^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta) \\ &= \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots} \frac{a^l}{r^{l+1}}\left(1 - \left(\frac{r}{b}\right)^{2l+1}\right) P_l(\cos\theta)\end{aligned}$$

As above, only the $l = 2$ term survives when we take the limit $a \to 0$

$$\Phi \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \left(\frac{r}{b}\right)^5\right) P_2(\cos\theta) = \frac{Q}{4\pi\epsilon_0 r^3}\left(1 - \left(\frac{r}{b}\right)^5\right)(3\cos^2\theta - 1)$$