Physics 505 **Fall 2007**

Homework Assignment #3 — Solutions

Textbook problems: Ch. 2: 2.14, 2.15, 2.22, 2.23

2.14 A variant of the preceeding two-dimensional problem is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.

a) Solve by means of the series solution (2.71) and show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}$$

The general series solution for the two-dimensional problem in polar coordinates is given by (2.71)

$$\Phi(\rho,\phi) = a_0 + b_0\log\rho + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n) + b_n\rho^{-n}\sin(n\phi+\beta_n)$$

Since we are interested in the interior solution, we demand that the potential remains finite at $\rho = 0$. This indicates that the b_n coefficients must all vanish. We are thus left with

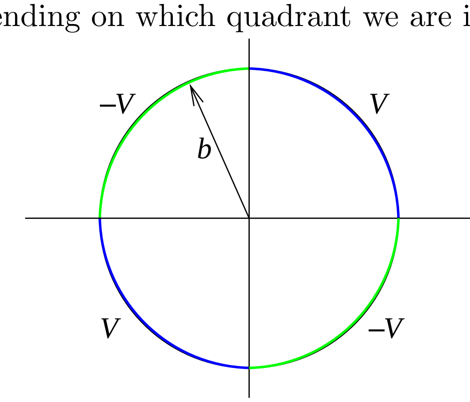
$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n)$$

which we may choose to rewrite as

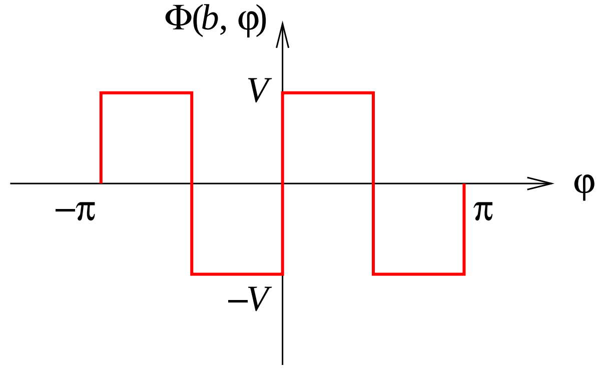
$$\Phi(\rho,\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty}[A_k\rho^k\cos(k\phi) + B_k\rho^k\sin(k\phi)] \tag{1}$$

This form of the series is supposed to be reminiscent of a Fourier series.

The boundary condition for this problem is that the potential at $\rho = b$ is either $+V$ or $-V$, depending on which quadrant we are in



This can be plotted as a function of ϕ



It should be obvious that $\Phi(b,\phi)$ is an odd function of ϕ. As a result, we immediately deduce that the A_k Fourier coefficients in (1) must vanish, leaving us with

$$\Phi(\rho,\phi)=\sum_{k=1}^{\infty} B_k\rho^k\sin(k\phi) \tag{2}$$

On the interior surface of the conducting cylinder, this reads

$$\Phi(b,\phi)=\sum_{k=1}^{\infty} B_k b^k\sin(k\phi)$$

where $\Phi(b,\phi)$ is given by the figure above. In particular, we see that the quantities $B_k b^k$ are explicitly the Fourier expansion coefficients of a square wave with period π (which is half the usual 2π period). As a result, we may simply look up the standard Fourier expansion of the square wave and map it to this present problem. Alternatively, it is straightforward to calculate the coefficients directly

$$\begin{aligned}
B_k b^k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(b,\phi)\sin(k\phi)d\phi \\
&= \frac{V}{\pi}\left(\int_{-\pi}^{-\pi/2}-\int_{-\pi/2}^{0}+\int_{0}^{\pi/2}-\int_{\pi/2}^{\pi}\right)\sin(k\phi)d\phi \\
&= \frac{V}{k\pi}\left(-\cos(k\phi)\Big|_{-\pi}^{-\pi/2}+\cos(k\phi)\Big|_{-\pi/2}^{0}-\cos(k\phi)\Big|_{0}^{\pi/2}-\cos(k\phi)\Big|_{\pi/2}^{\pi}\right) \\
&= \frac{2V}{k\pi}\left(1-2\cos\Big(\frac{k\pi}{2}\Big)+\cos(k\pi)\right) \\
&= \frac{8V}{k\pi} \qquad k=2,6,10,14,\ldots \qquad (ie\ k=4n+2)
\end{aligned}$$

Substituting $B_k=8V/k\pi b^k$ into (2) and usiing $k=4n+2$ then gives

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1} \tag{3}$$

b) Sum the series and show that

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin 2\phi}{b^4-\rho^4}\right)$$

This series is easy to sum if we work with complex variables. Since $\sin\theta$ is the imaginary part of $e^{i\theta}$, we write (3) as

$$\begin{aligned}\Phi(\rho,\phi) &= \frac{4V}{\pi}\Im\sum_{n=0}^{\infty}\frac{(\rho/b)^{4n+2}e^{(4n+2)i\phi}}{2n+1} \\ &= \frac{4V}{\pi}\Im\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1} = \frac{4V}{\pi}\Im\left(z+\tfrac{1}{3}z^3+\tfrac{1}{5}z^5+\cdots\right)\end{aligned} \tag{4}$$

where

$$z \equiv \frac{\rho^2}{b^2}e^{2i\phi} \tag{5}$$

Now recall that the Taylor series expansion for $\log(1+z)$ is given by

$$\log(1+z) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}z^k = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \tfrac{1}{5}z^5 - \cdots$$

We may eliminate the even powers of z by taking the difference between $\log(1+z)$ and $\log(1-z)$. This allows us to derive the series expression

$$z + \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \cdots = \frac{1}{2}\log\frac{1+z}{1-z}$$

Substituting this into (4) gives

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\Im\log\frac{1+z}{1-z} = \frac{2V}{\pi}\arg\frac{1+z}{1-z}$$

Since $\arg(x+iy) = \tan^{-1}(y/x)$, a bit of algebra gives

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\Im z}{1-|z|^2}\right)$$

Using the expression for z given in (5), we finally obtain

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2(\rho^2/b^2)\sin 2\phi}{1-(\rho^2/b^2)^2}\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2\sin 2\phi}{b^4-\rho^4}\right) \tag{6}$$

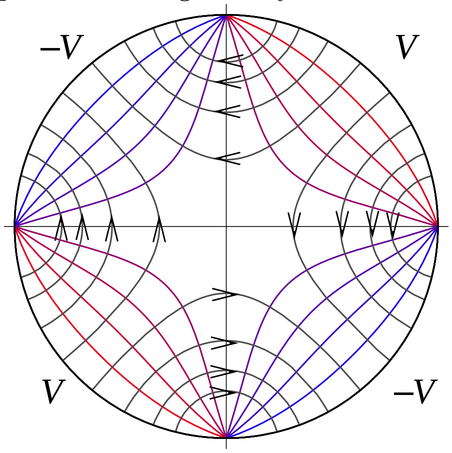
c) Sketch the field lines and equipotentials.

The equipotentials correspond to $\phi(\rho,\phi) = \Phi_0$. To see what this looks like, we may invert (6) to solve for b as a function of ϕ at fixed Φ_0. The result is

$$\frac{2\rho^2 b^2 \sin 2\phi}{b^4 - \rho^4} = \tan\left(\frac{\pi\Phi_0}{2V}\right)$$

$$\Rightarrow \qquad (\rho/b)^2 = -\frac{\sin 2\phi}{\tan(\pi\Phi_0/2V)} + \sqrt{1 + \frac{\sin^2 2\phi}{\tan^2(\pi\Phi_0/2V)}}$$

A plot of the equipotentials is given by



where we have also shown the electric field lines (the curves with arrows).

2.15 *a*) Show that the Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1$, $0 \le y \le 1$, has an expansion

$$G(x, y; x', y') = 2\sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -4\pi\delta(y' - y) \quad \text{and} \quad g_n(y, 0) = g_n(y, 1) = 0$$

We start by recalling the the Green's function is defined by

$$(\partial_{x'}^2 + \partial_{y'}^2) G(x, y; x', y') = -4\pi\delta(x' - x)\delta(y' - y) \tag{7}$$

Although this is symmetric in x' and y', the problem suggests that we begin by expanding in x' (and also x). This of course breaks the symmetry in the *expanded* form of the Green's function by treating x' somewhat differently. Nevertheless $G(x, y; x', y')$ is unique for the given boundary conditions; it just may admit different expansions, and we are free to choose whatever expansion is the most convenient.

Given the boundary condition that G vanishes for $x' = 0$ and $x' = 1$, this suggests an expansion in a Fourier sine series

$$G(x, y; x', y') = \sum_{n=1}^{\infty} f_n(x, y; y') \sin(n\pi x')$$

Substituting this into (7) then gives

$$\sum_{n=1}^{\infty} (\partial_{y'}^2 - n^2\pi^2) f_n(x, y; y') \sin(n\pi x') = -4\pi\delta(x' - x)\delta(y' - y) \tag{8}$$

However this is not particularly useful (yet), since the $\delta(x' - x)$ on the right hand side does not match with the Fourier sine series on the left. We can get around this by invoking the completeness relation for the sine series

$$\sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') = \tfrac{1}{2}\delta(x - x')$$

By replacing the delta function in (8) by this sum, we end up with

$$\sum_{n=1}^{\infty} (\partial_{y'}^2 - n^2\pi^2) f_n(x, y; y') \sin(n\pi x') = -8\pi\delta(y' - y) \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') \tag{9}$$

Matching left and right sides of the Fourier sine series indicates that the x behavior of $f_n(x, y; y')$ must be given by $\sin(n\pi x)$. Putting in a factor of two for convenience

$$f_n(x, y; y') = 2g_n(y, y') \sin(n\pi x)$$

finally motivates the expansion

$$G(x, y; x', y') = 2 \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

When this is inserted into (9), we match the x and x' behavior perfectly, and we are left with an equation in y'

$$(\partial_{y'}^2 - n^2\pi^2) g_n(y, y') = -4\pi\delta(y' - y) \tag{10}$$

The boundary conditions are that G vanishes at $y' = 0$ and $y' = 1$. Hence we must also demand $g_n(y, 0) = g_n(y, 1) = 0$.

b) Taking for $g_n(y, y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.

To find the Green's function for (10), we begin with the solution to the homogeneous equation $(\partial_{y'}^2 - n^2\pi^2)g_n(y,y') = 0$. This clearly has exponential solutions $e^{\pm n\pi y'}$, or equivalently $\sinh(n\pi y')$ and $\cosh(n\pi y')$. As a result, we can write the Green's function as

$$g_n(y,y') = \begin{cases} g_< \equiv a_<\sinh(n\pi y') + b_<\cosh(n\pi y') & y' < y \\ g_> \equiv a_>\sinh(n\pi y') + b_>\cosh(n\pi y') & y' > y \end{cases} \tag{11}$$

We wish to solve for the four constants $a_<, b_<, a_>, b_>$ given the boundary conditions $g_n(y,0) = 0$, $g_n(y,1) = 0$ and the continuity and jump conditions

$$g_> = g_< \qquad \partial_{y'}g_> = \partial_{y'}g_< - 4\pi \qquad \text{when} \quad y' = y$$

We start with the boundary conditions. For $g_<$ to vanish at $y' = 0$ we must take the sinh solution, while for $g_>$ to vanish at $y' = 1$ we end up with $a_>\sinh(n\pi) + b_>\cosh(n\pi) = 0$ or $b_> = -a_>\tanh(n\pi)$. Thus

$$g_n(y,y') = \begin{cases} a_<\sinh(n\pi y') & y' < y \\ a_>[\sinh(n\pi y') - \tanh(n\pi)\cosh(n\pi y')] & y' > y \end{cases} \tag{12}$$

The continuity and jump conditions yield the system of equations

$$\begin{pmatrix} \sinh(n\pi y) & -\sinh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi y) & -\cosh(n\pi y) + \tanh(n\pi)\sinh(n\pi y) \end{pmatrix}\begin{pmatrix} a_< \\ a_> \end{pmatrix} = \begin{pmatrix} 0 \\ 4/n \end{pmatrix}$$

which is solved by

$$\begin{aligned} \begin{pmatrix} a_< \\ a_> \end{pmatrix} &= -\frac{4}{n\tanh(n\pi)}\begin{pmatrix} \sinh(n\pi y) - \tanh(n\pi)\cosh(n\pi y) \\ \sinh(n\pi y) \end{pmatrix} \\ &= -\frac{4}{n\sinh(n\pi)}\begin{pmatrix} \cosh(n\pi)\sinh(n\pi y) - \sinh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi)\sinh(n\pi y) \end{pmatrix} \end{aligned}$$

Inserting this into (12) gives

$$\begin{aligned} g_n(y,y') = {} & \frac{4}{n\sinh(n\pi)} \\ & \times \begin{cases} \sinh(n\pi y')[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)] & y' < y \\ \sinh(n\pi y)[\sinh(n\pi)\cosh(n\pi y') - \cosh(n\pi)\sinh(n\pi y')] & y' > y \end{cases} \end{aligned}$$

This is simplified by noting

$$\sinh[n\pi(1-y)] = \sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)$$

and by using the definition $y_< = \min(y, y')$ and $y_> = \max(y, y')$. The result is

$$g_n(y, y') = \frac{4}{n\sinh(n\pi)}\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

which yields

$$G(x, y; x', y') = \sum_n \frac{8}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

Alternatively, instead of using (11), note that we can automatically solve the boundary conditions $g_n(y, 0) = g_n(y, 1) = 0$ by writing

$$g_n(y, y') = \begin{cases} g_< \equiv a_< \sinh(n\pi y') & y' < y \\ g_> \equiv a_> \sinh[n\pi(1-y')] & y' > y \end{cases}$$

Solving the continuity and jump conditions then gives directly

$$a_< = \frac{4}{n}\frac{\sinh[n\pi(1-y)]}{\sinh(n\pi)}, \qquad a_> = \frac{4}{n}\frac{\sinh(n\pi y)}{\sinh(n\pi)}$$

so that

$$g_n(y, y') = \frac{4}{n\sinh(n\pi)}\begin{cases} \sinh[n\pi(1-y)]\sinh(n\pi y') & y' < y \\ \sinh(n\pi y)\sinh[n\pi(1-y')] & y' > y \end{cases}$$

which is the same result as above. Finally, we note that the one-dimensional Green's function $g_n(y, y')$ can also be obtained through Sturm-Liouville theory as

$$g_n(y, y') = -\frac{1}{A}u(y_<)v(y_>)$$

where $u(y')$ and $v(y')$ are solutions to the homogeneous equation satisfying boundary conditions at $y' = 0$ and $y' = 1$, respectively. Here A is a constant given by $W(u, v) = A/p$ where W is the Wronskian, and the self-adjoint differential operator is

$$L = \frac{d}{dy'}p(y')\frac{d}{dy'} + q(y')$$

2.22 *a*) For the example of oppositely charged conducting hemispherical shells separated by a tiny gap, as shown in Figure 2.8, show that the interior potential ($r < a$) on the z axis is

$$\Phi_{\text{in}}(z) = V\frac{a}{z}\left[1 - \frac{(a^2 - z^2)}{a\sqrt{a^2 + z^2}}\right]$$

Find the first few terms of the expansion in powers of z and show that they agree with (2.27) with the appropriate substitutions.

As we have seen, the Green's function for the interior conducting sphere problem is equivalent to that for the exterior problem. The only difference we need to account for is that, for the interior problem, the outward pointing normal indeed points away from the center of the sphere. This indicates that the interior potential may be expressed as

$$\Phi(r.\Omega) = \frac{1}{4\pi}\int \Phi(a,\Omega')\frac{a(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}d\Omega'$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

In fact, introducing the absolute value $|a^2-r^2|$, it is easy to see that the expression

$$\Phi(r,\Omega) = \frac{a|a^2-r^2|}{4\pi}\int \frac{\Phi(a,\Omega')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}d\Omega'$$

is valid for both the interior and the exterior problem.

For the oppositely charged hemisphere problem, this integral takes the form

$$\begin{aligned}\Phi(r,\Omega) = \frac{Va|a^2-r^2|}{4\pi}&\int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta') \\ &\left[(r^2+a^2-2ar\cos\gamma)^{-3/2} - (r^2+a^2+2ar\cos\gamma)^{-3/2}\right]\end{aligned}$$

This simplifies on the z axis, where $\theta=0$ implies $\cos\gamma=\cos\theta'$. We find

$$\begin{aligned}\Phi(z) &= \frac{Va|a^2-z^2|}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta') \\ &\qquad \left[(z^2+a^2-2az\cos\theta')^{-3/2} - (z^2+a^2+2az\cos\theta')^{-3/2}\right] \\ &= \frac{V|a^2-z^2|}{2z}\left[(z^2+a^2-2az\cos\theta')^{-1/2} - (z^2+a^2+2az\cos\theta')^{-1/2}\right]_0^1 \\ &= \frac{V|a^2-z^2|}{2z}\left(\frac{1}{|z-a|}+\frac{1}{|z+a|}-\frac{2}{\sqrt{z^2+a^2}}\right) \\ &= \frac{V}{z}\left(\max(a,z) - \frac{|a^2-z^2|}{\sqrt{z^2+a^2}}\right)\end{aligned} \tag{13}$$

For the interior, $z<a$, this may be rewritten as

$$\Phi_{\text{in}}(z) = V\frac{a}{z}\left(1-\frac{a^2-z^2}{a\sqrt{a^2+z^2}}\right) \tag{14}$$

while for the exterior, $z > a$, this becomes

$$\Phi_{\text{out}}(z) = V\left(1 - \frac{z^2 - a^2}{z\sqrt{a^2 + z^2}}\right)$$

The interior solution, (14), may be expanded for $z \approx 0$. The result is

$$\Phi_{\text{in}}(z) = \frac{3V}{2}\left(\frac{z}{a}\right)\left[1 - \frac{7}{12}\left(\frac{z}{a}\right)^2 + \frac{11}{24}\left(\frac{z}{a}\right)^4 - \cdots\right] \tag{15}$$

This may be compared with the exterior solution (2.27)

$$\begin{aligned} \Phi_{\text{out}}(r, \theta) &= \frac{3V}{2}\left(\frac{a}{r}\right)^2\left[P_1(\cos\theta) - \frac{7}{12}\left(\frac{a}{r}\right)^2 P_3(\cos\theta) + \cdots\right] \\ \Rightarrow \quad \Phi_{\text{out}}(z) &= \frac{3V}{2}\left(\frac{a}{z}\right)^2\left[1 - \frac{7}{12}\left(\frac{a}{z}\right)^2 + \cdots\right] \end{aligned}$$

This demonstrates that the expansion coefficients agree, and that in fact the interior and exterior expressions are identical up to the substitution

$$\text{in:} \quad \left(\frac{r}{a}\right)^l \quad \leftrightarrow \quad \text{out:} \quad \left(\frac{a}{r}\right)^{l+1}$$

b) From the result of part a and (2.22), show that the radial electric field on the positive z axis is

$$E_r(z) = \frac{Va^2}{(z^2 + a^2)^{3/2}}\left(3 + \frac{a^2}{z^2}\right)$$

for $z > a$, and

$$E_r(z) = -\frac{V}{a}\left[\frac{3 + (a/z)^2}{(1 + (z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right]$$

for $|z| < a$. Show that the second form is well behaved at the origin, with the value, $E_r(0) = -3V/2a$. Show that at $z = a$ (north pole inside) it has the value $-(\sqrt{2} - 1)V/a$. Show that the radial field at the north pole outside has the value $\sqrt{2}V/a$.

The radial electric field on the positive z axis is given by

$$E_r(z) = -\frac{\partial}{\partial z}\Phi(z)$$

Rewriting the potential $\Phi(z)$ in (13) as

$$\Phi(z) = V\left(\max(a/z, 1) - \frac{|a^2 - z^2|}{z\sqrt{z^2 + a^2}}\right)$$

we find

$$E_r(z) = \begin{cases} -V\left(-\dfrac{a}{z^2} + \dfrac{a^2(a^2+3z^2)}{z^2(z^2+a^2)^{3/2}}\right) & z < a \\ -V\left(-\dfrac{a^2(a^2+3z^2)}{z^2(z^2+a^2)^{3/2}}\right) & z > a \end{cases}$$

This may be simplified to read

$$E_r(z) = \begin{cases} -\dfrac{V}{a}\left(\dfrac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \dfrac{a^2}{z^2}\right) & z < a \\ \dfrac{V}{a}\dfrac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} & z > a \end{cases} \tag{16}$$

which is the desired result. As $z \to 0$, we may Taylor expand the interior solution to obtain

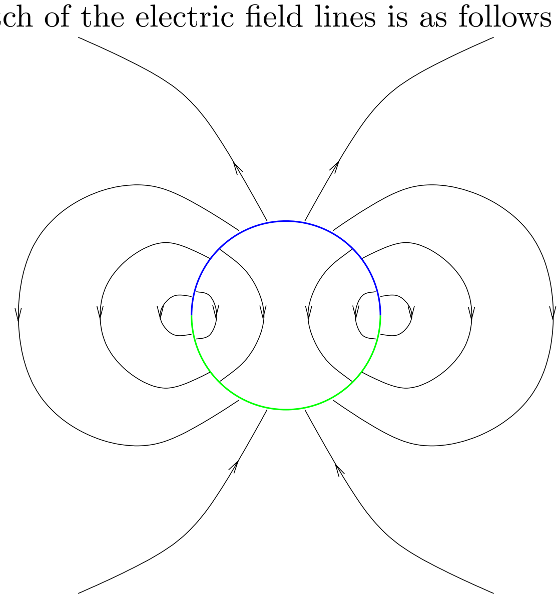
$$E_r(z) = -\frac{3V}{2a}\left[1 - \frac{7}{4}\left(\frac{z}{a}\right)^2 + \frac{55}{24}\left(\frac{z}{a}\right)^4 - \cdots\right]$$

Hence $E_r(0) = -3V/2a$. Note that this result could have been obtained directly by differentiating (15). Finally, the value of the radial electric field at $z = a_-$ (immediately inside) and $z = a_+$ (immediately outside) may be obtained from (16)

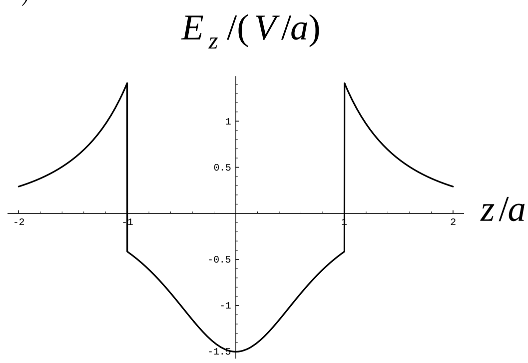
$$E_r(a_\pm) = \begin{cases} -\dfrac{V}{a}(\sqrt{2}-1) & z = a_- \\ \dfrac{V}{a}\sqrt{2} & z = a_+ \end{cases}$$

c) Make a sketch of the electric field lines both inside and outside the conducting hemispheres, with directions indicated. Make a *plot* of the radial electric field along the z axis from $z = -2a$ to $z = +2a$.

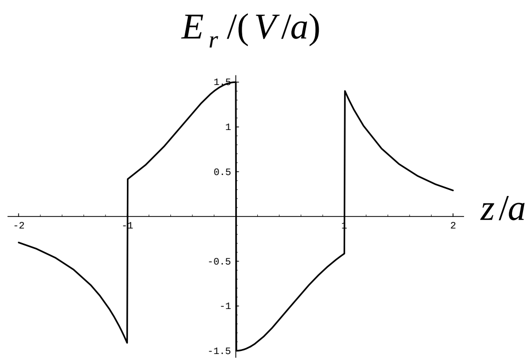
A rough sketch of the electric field lines is as follows



Note that the field lines are not necessarily continuous from the inside to the outside of the hemispheres. The z component of the electric field along the z axis is given by (16)



Note that E_z is positive (pointed upwards) outside the sphere, and negative (pointed downwards) inside the sphere. By symmetry, E_z is the only non-vanishing component of the electric field along the axis. The *radial* or r component of the electric field, E_r, is the same as E_z on the $+z$ axis, but has the opposite sign on the $-z$ axis



2.23 A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

The potential may be obtained by superposition

$$\Phi = \Phi_{\text{top}} + \Phi_{\text{bottom}}$$

where Φ_{top} (Φ_{bottom}) is the solution for a hollow cube with the top (bottom) held at constant potential V and all other sides at zero potential. As we have seen, the series solution for Φ_{top} is given by

$$\Phi_{\text{top}} = \sum_{n,m} A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sinh\left(\frac{\sqrt{n^2 + n^2}\,\pi z}{a}\right)$$

where

$$A_{n,m} = \frac{4}{a^2 \sinh(\sqrt{n^2 + m^2}\,\pi)} \int_0^a dx \int_0^a dy\, V \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right)$$

Noting that

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) dx = -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a = \frac{a}{n\pi}(1-(-1)^n) = \frac{2a}{n\pi} \quad \text{for } n \text{ odd}$$

we have

$$A_{n,m} = \frac{16V}{nm\pi^2 \sinh(\sqrt{n^2+m^2}\,\pi)} \qquad n, m \text{ odd}$$

and hence

$$\begin{aligned}\Phi_{\text{top}} = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} & \frac{1}{nm \sinh(\sqrt{n^2+m^2}\,\pi)} \\ & \times \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sinh\Big(\frac{\sqrt{n^2+n^2}\,\pi z}{a}\Big)\end{aligned}$$

To obtain Φ_{bottom}, it is sufficient to realize that symmetry allows us to take $z \to a - z$. More precisely

$$\Phi_{\text{bottom}}(x, y, z) = \Phi_{\text{top}}(x, y, a - z)$$

As a result

$$\begin{aligned}\Phi &= \Phi_{\text{top}} + \Phi_{\text{bottom}} \\ &= \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \frac{1}{nm \sinh(\sqrt{n^2+m^2}\,\pi)} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \\ &\qquad \times \left[\sinh\Big(\frac{\sqrt{n^2+n^2}\,\pi z}{a}\Big) + \sinh\Big(\frac{\sqrt{n^2+m^2}\,\pi(a-z)}{a}\Big)\right]\end{aligned}$$

Note that this may be simplified using

$$\sinh\zeta + \sinh(\alpha - \zeta) = 2\sinh(\alpha/2)\cosh(\zeta - \alpha/2)$$

to read

$$\begin{aligned}\Phi = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} & \frac{1}{nm \cosh(\sqrt{n^2+m^2}\,\pi/2)} \\ & \times \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \cosh\Big(\frac{\sqrt{n^2+m^2}\,\pi(z-a/2)}{a}\Big)\end{aligned} \tag{17}$$

b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

At the center of the cube, $(x, y, z) = (a/2, a/2, a/2)$, the potential from (17) reads

$$\begin{aligned}\Phi(\text{center}) &= \frac{16V}{\pi^2} \sum_{n,m\ \text{odd}} \frac{\sin(n\pi/2)\sin(m\pi/2)}{nm\cosh(\sqrt{n^2+m^2}\,\pi/2)} \\ &= \frac{16V}{\pi^2} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i+1)(2j+1)\cosh(\sqrt{(2i+1)^2+(2j+1)^2}\,\pi/2)}\end{aligned}$$

Numerically, the first few terms in this series are given by

n	m	$\Phi_{n,m}/V$	running total
1	1	.347546	.347546
1	3	$-.007524$	
3	1	$-.007524$	.332498
3	3	.000460	.332958
1	5	.000215	
5	1	.000215	.333389
3	5	$-.000023$	
5	3	$-.000023$	.333343

This table indicates that we need to keep at least the first four terms to achieve accuracy to three significant figures. To this level of accuracy, we have

$$\Phi(\text{center}) \approx .333V$$

If we went to higher orders, it appears that the potential at the center is precisely

$$\Phi(\text{center}) = \tfrac{1}{3}V$$

which is the average value of the potential on the walls. In fact, we can prove (as in Problem 2.28) that the potential at the center of a regular polyhedron is equal to the average of the potential on the walls. Hence this value of $V/3$ is indeed exact.

c) Find the surface-charge density on the surface $z = a$.

For the surface-charge density on the inside top surface ($z = a$), we use

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}\bigg|_S = \epsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=a}$$

where the normal pointing away from the top conductor is $\hat{n} = -\hat{z}$. This is what accounts for the sign flip in the above. Substituting in (17) gives

$$\sigma = \frac{16\epsilon_0 V}{\pi a} \sum_{n,m\ \text{odd}} \frac{\sqrt{n^2+m^2}}{nm} \tanh(\sqrt{n^2+m^2}\,\pi/2) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$