Physics 505 **Fall 2007**

Homework Assignment #2 — Solutions

Textbook problems: Ch. 2: 2.2, 2.8, 2.10, 2.11

2.2 Using the method of images, discuss the problem of a point charge q *inside* a hollow, grounded, conducting sphere of inner radius a. Find

a) the potential inside the sphere;

Recall that, if the point charge is outside a grounded conducting sphere, the method of images gives

$$\Phi(\vec{x}\,) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-\vec{y}\,|} - \frac{a/y}{|\vec{x}-(a/y)^2\vec{y}\,|}\right) \tag{1}$$

where $y = |\vec{y}\,|$, and $\vec{y}$ specifies the location of the charge q. Although this expression was derived for $y > a$, we note that it is equally valid for $y < a$. After all, in both cases, the potential satisfies the same boundary condition, $\Phi(|\vec{x}\,| = a) = 0$. Hence, for q inside the hollow sphere, the potential is also given by (1). In this case, the physical charge is inside the sphere, while the image charge lies outside. The image charge has the opposite sign, and in this case its magnitude is greater than the physical charge.

b) the induced surface-charge density;

The induced surface-charge density is given by

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial n}\right|_S$$

where in this case the unit normal is pointing into the sphere. Although we can work out this expression with Φ given by (1), it is quicker to note that the result must the be same (up to a sign change) as that for a point charge outside the sphere. The difference in sign is due to the inward pointing normal in this case, as opposed to an outward pointing normal when the point charge is outside. The result is

$$\sigma = \frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-(a/y)^2}{(1+(a/y)^2-2(a/y)\cos\gamma)^{3/2}}$$

where γ is the angle between $\vec{x}$ and $\vec{y}$. Note that σ has the opposite sign as q. This is because the numerator in the above expression is actually negative for $y < a$. If desired, this sign can be made explicit by rewriting the above as

$$\sigma = -\frac{q}{4\pi a^2}\frac{1-(y/a)^2}{(1+(y/a)^2-2(y/a)\cos\gamma)^{3/2}}$$

By Gauss' law, the total charge induced on the inside surface of the conducting sphere must be $-q$. This can also be seen by integrating the surface-charge density

$$
\begin{aligned}
Q = \int \sigma a^2 d\Omega = 2\pi a^2 \int_{-1}^{1} \sigma \, d\cos\gamma \\
= -\frac{q}{2}(1-(y/a)^2) \int_{-1}^{1} \frac{d\cos\gamma}{(1+(y/a)^2 - 2(y/a)\cos\gamma)^{3/2}} \\
= -\frac{q}{2}\left(\frac{a}{y}\right)(1-(y/a)^2)\left(\frac{1}{|1-y/a|} - \frac{1}{|1+y/a|}\right) = -q
\end{aligned}
$$

where we have used the fact that $0 < y/a < 1$ when simplifying the absolute value quantities.

c) the magnitude and direction of the force acting on q.

The force acting on q is essentially the force between q and its image. Again, the magnitude of the force is given by the familiar expression

$$
F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3 \frac{1}{(1-(a/y)^2)^2}
$$

Since the charge q is attracted to its image, the direction of the force is given by $\hat{y}$. For $y < a$, it is convenient to rewrite the force as

$$
\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^3}\frac{\vec{y}}{(1-(y/a)^2)^2}
$$

This demonstrates that, for $y \ll a$, the force is linear, $\vec{F} \sim \vec{y}$. Because of the positive sign, however, this is the *opposite* of a restoring force. This demonstrates that the center of the conducting sphere is a point of unstable equilibrium for the charge q.

d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

If the sphere is at a fixed potential V, the potential inside the sphere is given by adding V to the potential Φ given in (1). The induced surface-charge density on the inside surface and the force are unchanged.

For the case where the sphere has a total charge Q on its inner and outer surfaces, we recall that the charge on the inner surface must still be $-q$ by Gauss' law. Thus the charge on the outer surface must be $Q + q$. This indicates that the potential of the sphere must be

$$
V = \frac{1}{4\pi\epsilon_0}\frac{Q+q}{a}
$$

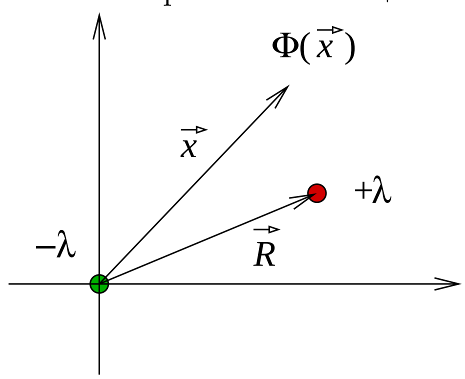
As a result, the potential inside the sphere is given by adding this V to (1). The induced surface-charge density on the inner surface and the force are unchanged. The surface-charge density on the outer surface is

$$\sigma_{\text{out}} = \frac{Q+q}{4\pi a^2}$$

2.8 A two-dimensional potential problem is defined by two straight parallel line charges separated by a distance R with equal and opposite linear charge densities λ and $-\lambda$.

a) Show by direct construction that the surface of constant potential V is a circular cylinder (circle in the transverse dimensions) and find the coordinates of the axis of the cylinder and its radius in terms of R, λ, and V.

For convenience, we place the $-\lambda$ line charge at the origin (in two dimensions). We then denote the position of the $+\lambda$ line charge by the vector displacement $\vec{R}$



By linear superposition, the potential is

$$\Phi(\vec{x}\,) = -\frac{\lambda}{2\pi\epsilon_0}\left(\log|\vec{x}-\vec{R}\,| - \log|\vec{x}\,|\right) = \frac{\lambda}{2\pi\epsilon_0}\log\frac{|\vec{x}\,|}{|\vec{x}-\vec{R}\,|}$$

We want to identify the equipotential surfaces $\Phi = V$, which is equivalent to

$$\frac{|\vec{x}\,|}{|\vec{x}-\vec{R}\,|} = \exp\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)$$

To avoid lengthy expressions, we define

$$\zeta \equiv \exp\left(-\frac{2\pi\epsilon_0 V}{\lambda}\right) \tag{2}$$

Then the equipotential surfaces are given by the locus of $\vec{x}$ such that

$$|\vec{x}\,-\vec{R}\,| = \zeta|\vec{x}|$$

To see that this defines a circle, we may square the expression to obtain

$$(1-\zeta^2)x^2 - 2\vec{x}\cdot\vec{R} + R^2 = 0$$

Completing the square in $\vec{x}$ gives

$$\left|\vec{x} - \frac{\vec{R}}{1-\zeta^2}\right|^2 = \frac{\zeta^2 R^2}{(1-\zeta^2)^2}$$

This is the equation of a circle

$$|\vec{x} - \vec{x}_0|^2 = \rho^2$$

where $\vec{x}_0$ is the center and ρ is the radius. Comparing with the above, we see that

$$\vec{x}_0 = \frac{\vec{R}}{1-\zeta^2} = \frac{\vec{R}}{1-\exp(-4\pi\epsilon_0 V/\lambda)} \tag{3}$$

and

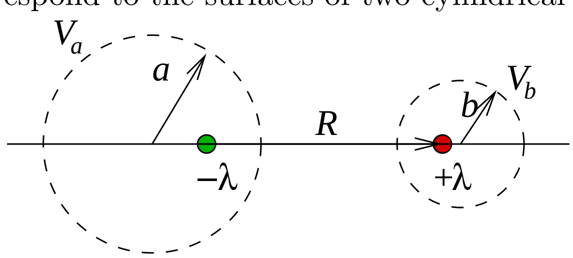
$$\rho = \frac{\zeta R}{|1-\zeta^2|} = \frac{R}{|\zeta - 1/\zeta|} = \frac{R}{2|\sinh(2\pi\epsilon_0 V/\lambda)|} \tag{4}$$

where we have substituted the expression (2) for ζ. Note that the center of the circle is either on the 'left' of the negative charge ($\vec{x}_0 = -c\vec{R}$ for $c > 1$) for $V < 0$ or on the 'right' of the positive charge ($\vec{x}_0 = c\vec{R}$ for $c > 1$) for $V > 0$. In particular, the center is always outside of the line segment joining the two charges.

b) Use the results of part a to show that the capacitance per unit length C of two right-circular cylindrical conductors, with radii a and b, separated by a distance $d > a + b$, is

$$C = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\dfrac{d^2 - a^2 - b^2}{2ab}\right)}$$

The result of part a indicates that the equipotential surfaces for two line charges are circles. In this case, we can pick two equipotential circles, and assume that they correspond to the surfaces of two cylindrical conductors.



Since the potential Φ is unique, we are guaranteed that this analogous problem of two infinitesimal line charges gives the correct solution for cylindrical conductors.

In order to obtain the capacitance per unit length

$$\frac{1}{C} = \frac{V_b - V_a}{\lambda} \tag{5}$$

we need appropriate expressions relating V_a/λ and V_b/λ to the geometry specified by a, b and d. These expressions may be obtained from (3) and (4) of part a. Starting with (3), the distance d is given by the sum of two terms

$$\begin{aligned}
d &= \frac{R}{1-\exp(-4\pi\epsilon_0 V_b/\lambda)} - \frac{R}{1-\exp(-4\pi\epsilon_0 V_a/\lambda)} \\
&= R\frac{\exp(-4\pi\epsilon_0 V_b/\lambda) - \exp(-4\pi\epsilon_0 V_a/\lambda)}{[1-\exp(-4\pi\epsilon_0 V_a/\lambda)][1-\exp(-4\pi\epsilon_0 V_b/\lambda)]} \\
&= \frac{R}{2}\frac{\sinh(2\pi\epsilon_0(V_a - V_b))}{\sinh(2\pi\epsilon_0 V_a/\lambda)\sinh(2\pi\epsilon_0 V_b/\lambda)}
\end{aligned}$$

where we note that $V_a < 0$, so that both terms in the first line are in fact positive. This may be rewritten as

$$\sinh\left(\frac{2\pi\epsilon_0(V_a - V_b)}{\lambda}\right) = \frac{2d}{R}\sinh\left(\frac{2\pi\epsilon_0 V_a}{\lambda}\right)\sinh\left(\frac{2\pi\epsilon_0 V_b}{\lambda}\right) \tag{6}$$

Also, using (4), we find the two relations

$$\sinh\left(\frac{2\pi\epsilon_0 V_a}{\lambda}\right) = -\frac{R}{2a}, \qquad \sinh\left(\frac{2\pi\epsilon_0 V_b}{\lambda}\right) = \frac{R}{2b} \tag{7}$$

between the potentials V_a and V_b and the radii a and b. In principle, all that remains is to solve (6) and (7) for the ratios V_a/λ and V_b/λ, and to insert this into the formula (5) for the capacitance per unit length. Unfortunately, these expressions are somewhat unwieldy. One way to proceed is to start with the hyperbolic trig identity

$$\cosh(\xi_a - \xi_b) = \cosh\xi_a\cosh\xi_b - \sinh\xi_a\sinh\xi_b$$

which may be rewritten as

$$\cosh(\xi_a - \xi_b) + \sinh\xi_a\sinh\xi_b = \cosh\xi_a\cosh\xi_b$$

Squaring this and manipulating the result gives an identity

$$\sinh^2(\xi_a - \xi_b) + 2\cosh(\xi_a - \xi_b)\sinh\xi_a\sinh\xi_b = \sinh^2\xi_a + \sinh^2\xi_b$$

which is equivalent to

$$\cosh(\xi_a - \xi_b) = \frac{1}{2}\left(\frac{\sinh\xi_a}{\sinh\xi_b} + \frac{\sinh\xi_b}{\sinh\xi_a} - \frac{\sinh^2(\xi_a - \xi_b)}{\sinh\xi_a\sinh\xi_b}\right)$$

The interesting feature of this expression is that the right hand side is fully determined in terms of a, b and d by elementary substitution of (6) and (7). In particular, we see that

$$\cosh\left(\frac{2\pi\epsilon_0(V_b - V_a)}{\lambda}\right) = \frac{1}{2}\left(-\frac{a}{b} - \frac{b}{a} + \frac{d^2}{ab}\right) = \frac{d^2 - a^2 - b^2}{2ab} \tag{8}$$

Note that it is also possible (and perhaps easier) to derive this identity by working backwards. Using (5), this immediately gives

$$\frac{1}{C} = \frac{1}{2\pi\epsilon_0} \cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right) \tag{9}$$

c) Verify that the result for C agrees with the answer in Problem 1.7 in the appropriate limit and determine the next nonvanishing order correction in powers of a/d and b/d.

Note that (9) may be rewritten as

$$\frac{1}{C} = \frac{1}{2\pi\epsilon_0} \log\left(\xi + \sqrt{\xi^2 - 1}\right)$$

where $\xi = (d^2 - a^2 - b^2)/(2ab)$. In the limit $d \gg a, b$, we have $\xi \gg 1$, so we may expand

$$\begin{aligned}
\frac{1}{C} &= \frac{1}{2\pi\epsilon_0}\left(\log(2\xi) - \frac{1}{4\xi^2} + \mathcal{O}(\frac{1}{\xi^4})\right) \\
&= \frac{1}{2\pi\epsilon_0}\left(\log\left(\frac{d^2}{ab}\right) + \log\left(1 - \frac{a^2+b^2}{d^2}\right) - \frac{a^2b^2}{d^4}\left(1 - \frac{a^2+b^2}{d^2}\right)^{-2} + \cdots\right) \\
&= \frac{1}{2\pi\epsilon_0}\left(\log\left(\frac{d^2}{ab}\right) - \frac{a^2+b^2}{d^2} + \cdots\right)
\end{aligned}$$

Inverting gives

$$C = \frac{\pi\epsilon_0}{\log(d/\sqrt{ab})}\left(1 + \frac{1}{\log(d/\sqrt{ab})}\frac{a^2+b^2}{2d^2} + \cdots\right)$$

This indeed agrees with the answer to Problem 1.7 in the large separation limit.

d) Repeat the calculation of the capacitance per unit length for two cylinders inside each other ($d < |b - a|$). Check the result for concentric cylinders ($d = 0$).

For the case of two cylinders inside each other, we take the potentials V_a and V_b to have the same sign. Assuming $V_a > V_b > 0$, the calculation is identical to that of part b above, except that there is no minus sign in the first term of (7). Formally, this is equivalent to taking $a \to -a$, which changes the overall sign of the right hand side of (8). The resulting formula for the capacitance per unit length is then

$$\frac{1}{C} = \frac{1}{2\pi\epsilon_0} \cosh^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right)$$

For concentric cylinders, we set $d = 0$ to obtain

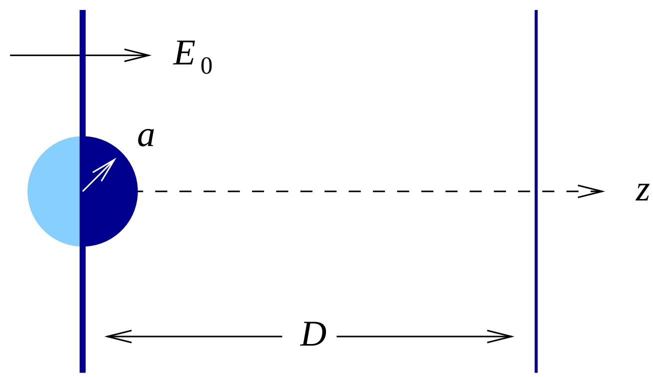
$$\frac{1}{C} = \frac{1}{2\pi\epsilon_0} \cosh^{-1}\left(\frac{a^2 + b^2}{2ab}\right) = \frac{\log|a/b|}{2\pi\epsilon_0}$$

This reproduces the familiar result from elementary treatments of cylindrical capacitors.

2.10 A large parallel plate capacitor is made up of two plane conducting sheets with separation D, one of which has a small hemispherical boss of radius a on its inner surface ($D \gg a$). The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is E_0.

a) Calculate the surface-charge densitites at an arbitrary point on the plane and on the boss, and sketch their behavior as a function of distance (or angle).

The way to approach this problem is to realize that the second conductor (the one without the boss) is kept far away from the region of interest (which is near the boss). Thus its only real purpose is to complete the capacitor and create a nearly uniform electric field E_0. As a result, this problem reduces to that of a conductor with a hemispherical boss in a uniform electric field. This, in turn, can be seen to be equivalent to half of the space of the conducting sphere in a uniform electric field setup.



Taking the conductor to be located at $z = 0$, the boss to be located at the origin, and the space between the plates to be $z > 0$, we end up with the (sphere in a uniform field) potential

$$\Phi = -E_0 z \left(1 - \frac{a^3}{r^3}\right) \qquad 0 < z < D$$

Of course, this result was obtained heuristically. Thus it would be useful to verify its correctness. To do so, we may easily show that the potential satisfies the appropriate conducting plate boundary conditions

$$\Phi(z = 0) = 0 \qquad \Phi(r = a) = 0$$

Of course, we should note that this is not an exact solution at the second conductor since $\Phi(z = D) = -E_0 D +$ (correction) is not absolutely constant. However, this is a perfectly reasonable solution to a high level of accuracy near the conductor with the boss.

Turning to the surface charge density, it is obtained by taking the normal derivative, $\sigma = -\epsilon_0 \partial\Phi/\partial n|_S$. On the plane, the normal direction is $\hat{z}$. Hence

$$\sigma_{\text{plane}} = -\epsilon_0 \partial_z \Phi\Big|_{z=0} = \epsilon_0 E_0 \left[1 - \frac{a^3}{r^3} + z\frac{3a^3 z}{r^5}\right]_{z=0} = \epsilon_0 E_0 \left(1 - \frac{a^3}{r^3}\right)$$

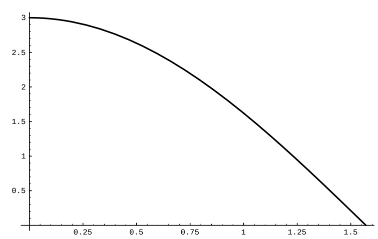
Note that the charge density vanishes at the location where the boss meets the plane ($r = a$).

On the boss, the normal direction is $\hat{r}$. Taking $z = r\cos\theta$, we obtain

$$\sigma_{\text{boss}} = -\epsilon_0 \partial_r \Phi\Big|_{r=a} = \epsilon_0 E_0 \left(1 + 2\frac{a^3}{r^3}\right)\cos\theta\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$$

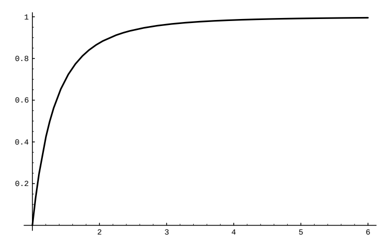
Again this vanishes at the joint between the boss and the plane (this is also consistent with the general theory of charge distribution near joints of conductors). The charge density σ may be plotted in units of $\epsilon_0 E_0$

boss



θ

plate



r/*a*

Note the different scales along the vertical axis. The charge density at the tip of the boss is three times that on the plate far away from the boss, and this is true for any size boss. Of course, far away from the plate, the relation $\sigma = \epsilon_0 E_0$ is a familiar one for parallel plate capacitors. In addition, however, this Gauss' law relation demonstrates the interesting fact that the electric field is three times as strong at the tip of the boss.

b) Show that the total charge on the boss has the magnitude $3\pi\epsilon_0 E_0 a^2$.

The charge on the boss is given by integrating

$$Q_{\text{boss}} = a^2 \int_0^{2\pi} d\phi \int_0^1 \sigma_{\text{boss}}\, d\cos\theta = 3\epsilon_0 E_0 a^2 (2\pi) \int_0^1 \cos\theta\, d\cos\theta = 3\pi\epsilon_0 E_0 a^2$$

c) If, instead of the other conducting sheet at a different potential, a point charge q is placed directly above the hemispherical boss at a distance d from its center, show that the charge induced on the boss is

$$q' = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right]$$

Taking away the second conductor (*i.e.* removing the uniform electric field) turns this into an image charge problem for a point charge near a conducting sphere. For the sphere by itself, a charge q at position d generates an image $-q(a/d)$ at location a^2/d. Starting from this, we introduce the conducting plane at $z = 0$. This gives additional image charges based on the reflection $z \to -z$. The images of the original charge and first image are thus $-q$ and $-d$ and $q(a/d)$ and $-a^2/d$. In other words

$$\Phi(\vec{x}\,) = kq\left(\frac{1}{|\vec{x} - d\hat{z}|} - \frac{a/d}{|\vec{x} - (a^2/d)\hat{z}|} - \frac{1}{|\vec{x} + d\hat{z}|} + \frac{a/d}{|\vec{x} + (a^2/d)\hat{z}|}\right) \tag{10}$$

The surface charge on the boss is given by $\sigma = -\epsilon_0 \hat{x} \cdot \vec{\nabla}\Phi|_{x=a}$, which has for the most part been calculated several times before in the spherical conductor examples. The result for (10) is

$$\begin{aligned}
\sigma &= -\epsilon_0 kq\left(\frac{d^2 - a^2}{a(d^2 + a^2 - 2ad\cos\theta)^{3/2}} - \frac{d^2 - a^2}{a(d^2 + a^2 + 2ad\cos\theta)^{3/2}}\right) \\
&= -\frac{q}{4\pi a}(d^2 - a^2)\left(\frac{1}{(d^2 + a^2 - 2ad\cos\theta)^{3/2}} - \frac{1}{(d^2 + a^2 + 2ad\cos\theta)^{3/2}}\right)
\end{aligned}$$

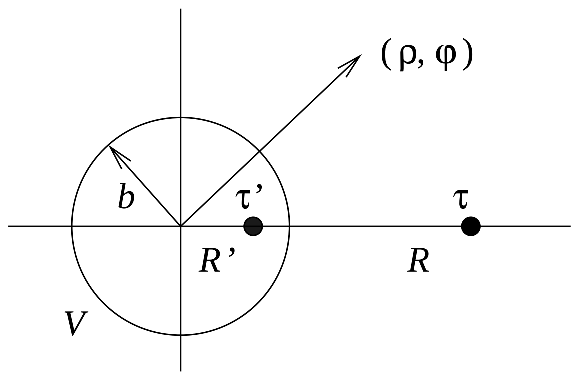
The total charge on the boss is given by integration

$$\begin{aligned}
Q_{\text{boss}} &= -\frac{q}{4\pi a}(d^2 - a^2)(2\pi a^2)\int_0^1 d\cos\theta\Bigg(\frac{1}{(d^2 + a^2 - 2ad\cos\theta)^{3/2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{(d^2 + a^2 + 2ad\cos\theta)^{3/2}}\Bigg) \\
&= -\frac{q}{2d}(d^2 - a^2)\left[(d^2 + a^2 - 2ad\cos\theta)^{-1/2} + (d^2 + a^2 + 2ad\cos\theta)^{1/2}\right]_0^1 \\
&= -\frac{q}{2d}(d^2 - a^2)\left(\frac{1}{d - a} + \frac{1}{d + a} - \frac{2}{(d^2 + a^2)^{1/2}}\right) \\
&= -q\left(1 - \frac{d^2 - a^2}{d(d^2 + a^2)^{1/2}}\right)
\end{aligned}$$

2.11 A line charge with linear charge density τ is placed parallel to, and a distance R away from, the axis of a conducting cylinder of radius b held at fixed voltage such that the potential vanishes at infinity. Find

a) the magnitude and position of the image charge(s);

We set up the system using polar coordinates as follows



We wish to determine the magnitude τ' and position R' of the image charge. Here we have assumed that only one image line charge is needed, and that by symmetry it falls along the line connecting the origin to the charge τ. For line charges, the potential may be written as

$$\begin{aligned}\Phi(\rho,\phi) &= -\frac{1}{2\pi\epsilon_0}\left(\tau\log|\vec{x}-\vec{R}\,| + \tau'\log|\vec{x}-\vec{R}'|\right)\\ &= -\frac{1}{4\pi\epsilon_0}\left(\tau\log(\rho^2+R^2-2\rho R\cos\phi) + \tau'\log(\rho^2+R'^2-2\rho R'\cos\phi)\right)\end{aligned} \tag{11}$$

We are fortunate that this problem specifies that the potential vanishes at infinity, $\Phi(\rho=\infty)=0$. This is because taking $\rho\to\infty$ in the above gives

$$\Phi(\rho\to\infty) \sim -\frac{\tau+\tau'}{2\pi\epsilon_0}\log\rho$$

and the only way for this to vanish is to choose $\tau'=-\tau$. Using this, the above expression (11) for the potential may be rewritten as

$$\Phi(\rho,\phi) = \frac{\tau}{4\pi\epsilon_0}\log\left(\frac{\rho^2+R'^2-2\rho R'\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right) \tag{12}$$

Any other value of τ' would lead to an unmanageable problem. Of course, we are not done yet, as we must also determine the location R' of the image charge. To do this, we impose the constant voltage boundary condition $\Phi(b,\phi)=V$, which translates to

$$\frac{b^2+R'^2-2bR'\cos\phi}{b^2+R^2-2bR\cos\phi} = \exp\left(\frac{4\pi\epsilon_0 V}{\tau}\right) \equiv \lambda^2 \tag{13}$$

where the last equality is taken as the definition of the constant λ. Multiplying out by the denominator and rearranging, this is equivalent to

$$(1-\lambda^2)b^2 + R'^2 - \lambda^2R^2 = 2b(R'-\lambda^2R)\cos\phi$$

Since we need this equation to hold for any angle ϕ along the cylinder, we see that both sides have to independently vanish. This leads to

$$R' = \lambda^2R, \qquad \lambda^2R^2 - R'^2 = (1-\lambda^2)b^2$$

which may be solved to give

$$\lambda = \frac{b}{R}, \qquad R' = \frac{b^2}{R}$$

This indicates that the image has charge τ' and location R' where

$$\tau' = -\tau, \qquad R' = \frac{b^2}{R}$$

Incidentally, from the definition of λ in (13), we see that the potential of the cylindrical conductor is

$$V = -\frac{\tau}{2\pi\epsilon_0}\log\left(\frac{R}{b}\right)$$

This is negative for $\tau > 0$.

b) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis), including the asymptotic form far from the cylinder;

Substituting $R' = b^2/R$ into (12) gives the potential

$$\Phi(\rho, \phi) = \frac{\tau}{4\pi\epsilon_0}\log\left(\frac{\rho^2 + b^4/R^2 - 2\rho(b^2/R)\cos\phi}{\rho^2 + R^2 - 2\rho R\cos\phi}\right) \tag{14}$$

For $\rho \gg R > b > 0$, this may be expanded to yield

$$\Phi = \frac{\tau}{2\pi\epsilon_0}\left[\frac{R^2 - b^2}{\rho R}\cos\phi + \frac{R^4 - b^4}{2\rho^2 R^2}\cos 2\phi + \frac{R^6 - b^6}{3\rho^3 R^3}\cos 3\phi + \mathcal{O}\left(\frac{1}{\rho^4}\right)\right]$$

Note that this is an expansion in harmonics of the form $\cos(n\phi)/\rho^n$

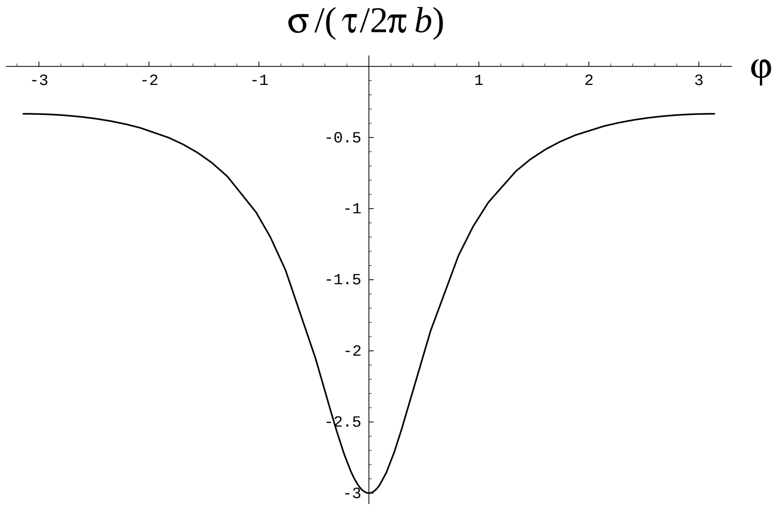
$$\Phi = \frac{\tau}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{R^{2n} - b^{2n}}{n(\rho R)^n}\cos(n\phi)$$

c) the induced surface-charge density, and plot it as a function of angle for $R/b = 2$, 4 in units of $\tau/2\pi b$;

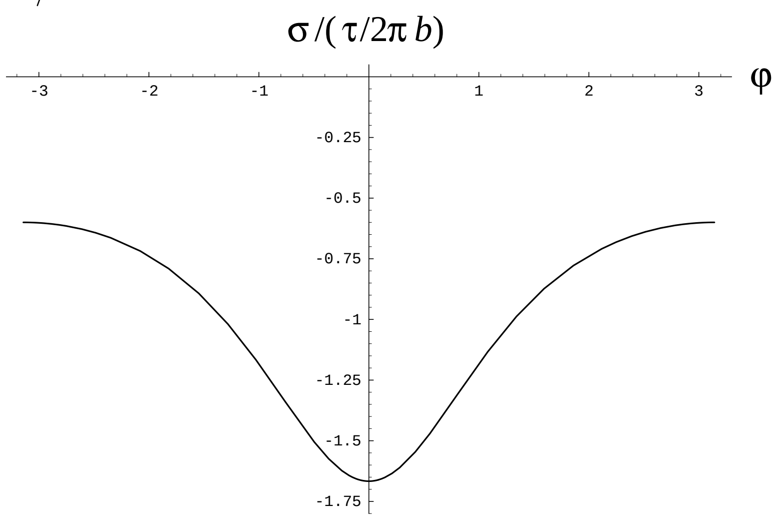
Using (14), we compute the surface-charge density

$$\begin{aligned}
\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} &= \frac{\tau}{2\pi}\left[\frac{\rho - R\cos\phi}{\rho^2 + R^2 - 2\rho R\cos\phi} - \frac{\rho - (b^2/R)\cos\phi}{\rho^2 + b^4/R^2 - 2\rho(b^2/R)\cos\phi}\right]_{\rho=b} \\
&= \frac{\tau}{2\pi}\frac{(b - R\cos\phi) - (R^2/b^2)(b - (b^2/R)\cos\phi)}{b^2 + R^2 - 2bR\cos\phi} \\
&= -\frac{\tau}{2\pi b}\frac{R^2 - b^2}{b^2 + R^2 - 2bR\cos\phi} \\
&= -\frac{\tau}{2\pi b}\frac{(R/b)^2 - 1}{(R/b)^2 + 1 - 2(R/b)\cos\phi}
\end{aligned}$$

For $R/b = 2$, the plot of the induced surface-charge density is as follows



while for $R/b = 4$ we have



The induced surface-charge density is greatest in magnitude on the side of the cylinder closest to the line charge. For $R/b = 4$, the line charge is further away from the cylinder, and the peak to $\phi = 0$ is less pronounced. Note that the total charge (per unit length) on the cylinder is given by

$$Q = \int_0^{2\pi} \sigma \, b d\phi = -\frac{\tau}{2\pi}(R^2 - b^2) \int_0^{2\pi} \frac{d\phi}{b^2 + R^2 - 2bR\cos\phi}$$

This integral may be performed by contour integration, using the substitution $z = e^{i\phi}$. The result is

$$\int_0^{2\pi} \frac{d\phi}{b^2 + R^2 - 2bR\cos\phi} = \frac{2\pi}{|R^2 - b^2|}$$

Hence we see that $Q = -\tau$ (independent of the ratio R/b), as expected.

d) the force per unit length on the line charge.

For a line charge of strength τ_1 in an electric field $E_2 = \tau_2/2\pi\epsilon_0 d$ created by a line charge of strength τ_2, the magnitude of the force is $F = \tau_1 E_2 = \tau_1\tau_2/2\pi\epsilon_0 d$, where d is the separation between the charges. In this case, the charge τ is attracted to its image of strength $-\tau$, and the separation is

$$d = R - R' = R - b^2/R = \frac{R^2 - b^2}{R}$$

As a result, the force per unit length is

$$F = \frac{1}{2\pi\epsilon_0}\frac{\tau^2}{b}\frac{bR}{R^2 - b^2}$$

and the line charge τ is attracted towards the center of the conducting cylinder.