

Homework Assignment #1 — Solutions

Textbook problems: Ch. 1: 1.5, 1.7, 1.11, 1.12

1.5 The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

We may obtain the charge distribution by computing $\rho = -\epsilon_0 \nabla^2 \Phi$. However, since Φ blows up as $r \rightarrow 0$, we must be a bit careful. We first consider $r > 0$

$$\begin{aligned} \rho = -\epsilon_0 \nabla^2 \Phi &= -\frac{q}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \left[e^{-\alpha r} \left(\frac{1}{r} + \frac{\alpha}{2} \right) \right] \\ &= \frac{q}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left[e^{-\alpha r} \left(1 + \alpha r + \frac{\alpha^2 r^2}{2} \right) \right] = -\frac{q\alpha^3}{8\pi} e^{-\alpha r} \end{aligned}$$

For $r \approx 0$, on the other hand, we may expand

$$\Phi = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{\alpha}{2} + \dots \right) \approx \frac{q}{4\pi\epsilon_0 r}$$

This is the potential of a point charge q at the origin. Hence the complete charge distribution can be written as

$$\rho = q\delta^3(r) - \frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

The first term corresponds to the proton charge, and the second to the negatively charged electron cloud in the $1s$ orbital around the proton.

We can additionally verify that the hydrogen atom is indeed neutral

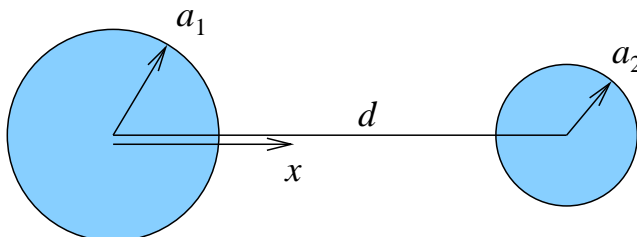
$$Q = \int \rho d^3x = q - \frac{q\alpha^3}{8\pi} \int_0^\infty e^{-\alpha r} 4\pi r^2 dr = q - \frac{q}{2} \Gamma(3) = 0$$

1.7 Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d , which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$C = \pi\epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}$$

where a is the geometrical mean of the two radii.

This is essentially a two-dimensional electrostatic problem. We choose the geometry to be



To calculate the capacitance, we use $C = Q/\Delta\Phi$ where C is the capacitance per unit length, Q ($-Q$) is the charge per unit length on the first (second) conductor, and $\Delta\Phi = \Phi_1 - \Phi_2$ is the potential difference between the conductors.

The potential outside a single cylindrical conductor located at the origin is given by the familiar expression

$$\Phi = -\frac{Q}{2\pi\epsilon_0} \log r$$

which may be obtained by integrating the electric field

$$\vec{E} = \frac{Q}{2\pi\epsilon_0} \frac{\hat{r}}{r}$$

(which in turn is derived by straightforward application of Gauss' law in integral form). When the radii are small compared to the separation, we may superpose the potentials for the two conductors to give

$$\Phi(x) \approx -\frac{Q}{2\pi\epsilon_0} [\log x - \log(d-x)]$$

which is valid for $a_1 \leq x \leq d - a_2$ (ie, for x between the two conductors). Note that this is not an exact result, since the exact potential should be constant on the surfaces of each conductor, while the above expression is not. The correction terms are of order a_1/d or a_2/d , and may be dropped in this large separation limit. We now compute the potential difference

$$\begin{aligned} \Delta\Phi = \Phi(a_1) - \Phi(d - a_2) &\approx \frac{Q}{2\pi\epsilon_0} [-\log a_1 + \log(d - a_1) + \log(d - a_2) - \log a_2] \\ &\approx \frac{Q}{2\pi\epsilon_0} \log \frac{(d - a_1)(d - a_2)}{a_1 a_2} \\ &\approx \frac{Q}{2\pi\epsilon_0} \log \frac{d^2}{a_1 a_2} \end{aligned}$$

where we have assumed $d \gg a_1$ and $d \gg a_2$. This gives the approximate expression for the capacitance

$$C = \frac{Q}{\Delta\Phi} \approx 2\pi\epsilon_0 \left(\log \frac{d^2}{a_1 a_2} \right)^{-1} = \pi\epsilon_0 \left(\log \frac{d}{a} \right)^{-1} \quad (1)$$

where $a = \sqrt{a_1 a_2}$ is the geometric mean of the two radii.

Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of 1.2×10^{-11} F/m if the separation of the wires was 0.5 cm? 1.5 cm? 5.0 cm?

This is one of the few problems where we will actually put in numbers. We first rewrite (1) as

$$a = d e^{-\pi\epsilon_0/C}$$

This show that, for a fixed capacitance per unit length, the relation between a and d is linear. For the above value of the capacitance, we obtain

$$a \approx 0.1d$$

Note that a is radius of the wire, while d is the separation. For the above numbers, the wire diameters and gauges are approximately

Separation	Diameter	AWG Gauge	Metric Gauge
0.5 cm	1 mm	18	10
1.5 cm	3 mm	9	30
5.0 cm	10 mm	000	100

Hopefully you agree that the metric wire gauges make much more sense.

- 1.11 Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

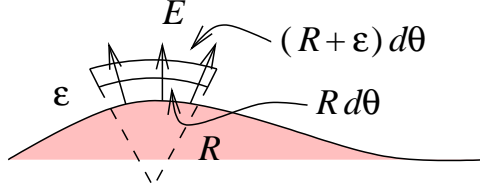
$$\frac{1}{E} \frac{\partial E}{\partial n} = - \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are the principal radii of curvature of the surface.

We actually use Gauss' law in integral form

$$\int_S \vec{E} \cdot \hat{n} da = 0$$

when there are no charges enclosed. Before considering the three-dimensional problem, consider the analogous situation in two dimensions



Take a curved Gaussian box next to the surface of the charged conductor at a point where the radius of curvature is R . Gauss' law then states

$$0 = \int_S \vec{E} \cdot \hat{n} da = E_{\text{top}} \Delta a_{\text{top}} - E_{\text{bottom}} \Delta a_{\text{bottom}} \quad (2)$$

where Δa_{top} and Δa_{bottom} are the areas of the top and bottom of the box, respectively. (There is no contribution from the sides of the box, because they are taken to be normal to the surface.) Using $\Delta a_{\text{top}} = (R + \epsilon)d\theta dz$ and $\Delta a_{\text{bottom}} = R d\theta dz$ gives

$$0 = E_{\text{top}}(R + \epsilon)d\theta dz - E_{\text{bottom}}R d\theta dz$$

which yields the relation

$$E_{\text{bottom}} = E_{\text{top}} \left(1 + \frac{\epsilon}{R}\right)$$

This allows us to calculate

$$\frac{\partial E}{\partial n} = \lim_{\epsilon \rightarrow 0} \frac{E_{\text{top}} - E_{\text{bottom}}}{\epsilon} = \lim_{\epsilon \rightarrow 0} -\frac{E_{\text{top}}}{R} = -\frac{E_{\text{top}}}{R}$$

Noting that E_{top} is the same as E when $\epsilon \rightarrow 0$, this may be rewritten as

$$\frac{1}{E} \frac{\partial E}{\partial n} = -\frac{1}{R} \quad (3)$$

which is the analogous two-dimensional expression.

Coming back to the three-dimensional problem, we use the same method as above. This time, however, the areas of the top and bottom of the Gaussian box are

$$\Delta a_{\text{top}} = (R_1 + \epsilon)(R_2 + \epsilon)d\Omega, \quad \Delta a_{\text{bottom}} = R_1 R_2 d\Omega$$

Substituting this into (2) gives

$$E_{\text{bottom}} = E_{\text{top}} \left(1 + \frac{\epsilon}{R_1}\right) \left(1 + \frac{\epsilon}{R_2}\right)$$

which in turn yields

$$\begin{aligned} \frac{\partial E}{\partial n} &= \lim_{\epsilon \rightarrow 0} \frac{E_{\text{top}} - E_{\text{bottom}}}{\epsilon} = \lim_{\epsilon \rightarrow 0} -E_{\text{top}} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{\epsilon}{R_1 R_2} \right) \\ &= -E_{\text{top}} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \end{aligned}$$

Rearranging this expression finally gives

$$\frac{1}{E} \frac{\partial E}{\partial n} = - \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (4)$$

Note that this reduces to the two-dimensional expression (3) in the cylindrical limit $R_2 \rightarrow \infty$.

It is worth noting that, when proving the above, the Gaussian box is actually taken to be infinitesimal. This suggests that we may use Gauss' law in differential form

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (\text{charge-free region})$$

Assuming the electric field is in the normal direction

$$\vec{E} = E \hat{n}$$

then gives

$$0 = \vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (E \hat{n}) = \hat{n} \cdot \vec{\nabla} E + E \vec{\nabla} \cdot \hat{n}$$

Using $\hat{n} \cdot \vec{\nabla} = \partial/\partial n$ and rearranging gives

$$\frac{1}{E} \frac{\partial E}{\partial n} = - \vec{\nabla} \cdot \hat{n}$$

This is a neat expression because a bit of geometry indicates that the divergence of the unit vector field is related to the principal radii of curvature of the associated surface by

$$\vec{\nabla} \cdot \hat{n} = \frac{1}{R_1} + \frac{1}{R_2}$$

Substituting this into the above then reproduces (4).

Finally, note that neither of the derivations of (4) actually require the physical existence of the conducting surface. Thus the expression (4) is valid in any charge-free region of space, provided we take E and \hat{n} to be the magnitude and direction of the electric field. In this case, R_1 and R_2 represent the principal radii of curvature of the equipotential surface corresponding to the electric field. With a little bit of thought, it is easy to see that the physical interpretation of (4) is that the electric field either gets weaker if the field lines diverge, or stronger if the field lines converge. (Note that, formally, R_1 and R_2 can be positive or negative.)

1.12 Prove *Green's reciprocity theorem*: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V , while Φ' is the potential due to another charge distribution ρ' and σ' , then

$$\int_V \rho \Phi' d^3x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_S \sigma' \Phi da$$

We start with Green's theorem for the potentials Φ and Φ'

$$\int_V (\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi) d^3x = \int_S \left(\Phi \frac{\partial \Phi'}{\partial n} - \Phi' \frac{\partial \Phi}{\partial n} \right) da$$

On the left hand side, we use

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho \quad \nabla^2 \Phi' = -\frac{1}{\epsilon_0} \rho'$$

This gives

$$\begin{aligned} \int_V (\rho \Phi' - \rho' \Phi) d^3x &= \epsilon_0 \int_S \left(\Phi \frac{\partial \Phi'}{\partial n} - \Phi' \frac{\partial \Phi}{\partial n} \right) da \\ &= \epsilon_0 \int_S (\Phi' E_{\perp} - \Phi E'_{\perp}) da \end{aligned} \tag{5}$$

where E_{\perp} is the normal electric field at the surface of the conducting surface, $E_{\perp} \equiv \hat{n} \cdot \vec{E} = -\partial\Phi/\partial n$. Since \hat{n} is an *outward* pointing normal, and since \vec{E} is the electric field on the interior of the surface, application of Gauss' law at the conducting surface gives

$$E_{\perp} = -\frac{1}{\epsilon_0} \sigma$$

Substituting this into (5) and rearranging then gives

$$\int_V \rho \Phi' d^3x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_S \sigma' \Phi da$$