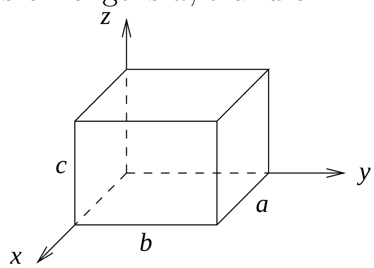
Practice Midterm — Solutions

This midterm will be a two hour open book, open notes exam. Do all three problems.

1. A rectangular box has sides of lengths a, b and c



a) For the Dirichlet problem in the interior of the box, the Green's function may be expanded as

$$G(x,y,z;x',y',z') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}(z,z') \sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}$$

Write down the appropriate differential equation that $g_{mn}(z,z')$ must satisfy.

Note that $\sin kx$ satisfies the completeness relation

$$\sum_{m=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a} = \frac{a}{2}\delta(x-x')$$

Hence the Green's function equation

$$\nabla^2_{x'} G(\vec{x},\vec{x}\,') = -4\pi\delta^3(\vec{x}-\vec{x}\,')$$

has an expansion

$$\sum_{m,n}\nabla^2_{x'}\left(g_{mn}(z,z')\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}\right)$$
$$= -4\pi\delta(z-z')\frac{4}{ab}\sum_{m,n}\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}$$

Working out the x' and y' derivatives on the left-hand side yields

$$\sum_{m,n}\left[\frac{d^2}{dz'^2}-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]g_{mn}(z,z')\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}$$
$$= -\frac{16\pi}{ab}\delta(z-z')\sum_{m,n}\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}$$

However, since $\sin kx$ forms an orthogonal basis, each term in this sum must vanish by itself. This results in the differential equation

$$\left(\frac{d^2}{dz'^2} - \gamma_{mn}^2\right) g_{mn}(z,z') = -\frac{16\pi}{ab}\delta(z-z') \tag{1}$$

where $\gamma_{mn} = \pi\sqrt{(m/a)^2 + (n/b)^2}$ is given in part c). Note that the Fourier sine expansion automatically satisfies Dirichlet boundary conditions for x and y. The remaining boundary condition is that $g_{mn}(z,z')$ vanishes whenever z or z' is equal to 0 or c.

b) Solve the Green's function equation for $g_{mn}(z,z')$ subject to Dirichlet boundary conditions and write down the result for $G(x,y,z;x',y',z')$.

We may solve the Green's function equation (1) by first noting that the homogeneous equation is of the form

$$g''(z') - \gamma_{mn}^2 g(z') = 0$$

This is a second-order linear equation with constant coefficients admitting the familiar solution

$$g(z') = Ae^{\gamma_{mn}z'} + Be^{-\gamma_{mn}z'}$$

However, we want $g(z') = 0$ when $z' = 0$ or $z' = c$. This motivates us to write out the solutions

$$\begin{aligned} u(z') &= \sinh \gamma_{mn} z' \qquad && 0 < z' < z \\ v(z') &= \sinh[\gamma_{mn}(c - z')] \qquad && z < z' < c \end{aligned}$$

The Green's function solution is then given by

$$g_{mn}(z,z') = \begin{cases} Au(z') & z' < z \\ Bv(z') & z' > z \end{cases}$$

The matching conditions

$$g_< = g_>, \qquad \frac{d}{dz'}g_< = \frac{d}{dz'}g_> + \frac{16\pi}{ab}$$

then give the system

$$\begin{aligned} A\sinh\gamma_{mn}z &= B\sinh[\gamma_{mn}(c-z)], \\ A\cosh\gamma_{mn}z &= -B\cosh[\gamma_{mn}(c-z)] + \frac{16\pi}{ab\gamma_{mn}} \end{aligned}$$

which may be solved to yield

$$\begin{aligned} A &= \frac{16\pi\sinh[\gamma_{mn}(c-z)]}{ab\gamma_{mn}(\cosh\gamma_{mn}z\sinh[\gamma_{mn}(c-z)] + \sinh\gamma_{mn}z\cosh[\gamma_{mn}(c-z)])} \\ &= \frac{16\pi\sinh[\gamma_{mn}(c-z)]}{ab\gamma_{mn}\sinh\gamma_{mn}c} = \frac{16\pi}{ab\gamma_{mn}\sinh\gamma_{mn}c}v(z) \end{aligned}$$

and

$$
\begin{aligned}
B &= \frac{16\pi \sinh \gamma_{mn} z}{ab\gamma_{mn}(\cosh \gamma_{mn} z \sinh[\gamma_{mn}(c-z)] + \sinh \gamma_{mn} z \cosh[\gamma_{mn}(c-z)])} \\
&= \frac{16\pi \sinh \gamma_{mn} z}{ab\gamma_{mn} \sinh \gamma_{mn} c} = \frac{16\pi}{ab\gamma_{mn} \sinh \gamma_{mn} c} u(z)
\end{aligned}
$$

As a result, the full Green's function solution is then given by

$$
\begin{aligned}
g_{mn}(z, z') &= \frac{16\pi}{ab\gamma_{mn} \sinh \gamma_{mn} c} u(z_<) v(z_>) \\
&= \frac{16\pi}{ab\gamma_{mn} \sinh \gamma_{mn} c} \sinh \gamma_{mn} z_< \sinh \gamma_{mn}[(c - z_>)]
\end{aligned}
$$

Hence

$$
\begin{aligned}
G(\vec{x}, \vec{x}\,') = \frac{16\pi}{ab} \sum_{m,n} \frac{1}{\gamma_{mn} \sinh \gamma_{mn} c} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \\
\times \sinh \gamma_{mn} z_< \sinh \gamma_{mn}[(c - z_>)]
\end{aligned} \tag{2}
$$

c) Consider the boundary value problem where the potential on top of the box is $\Phi(x, y, c) = V(x, y)$ while the potential on the other five sides vanish. Using the Greens' function obtained above, show that the potential may be written as

$$
\Phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \gamma_{mn} z
$$

where $\gamma_{mn} = \pi\sqrt{(m/a)^2 + (n/b)^2}$ and

$$
A_{mn} = \frac{4}{ab \sinh \gamma_{mn} c} \int_0^a dx \int_0^b dy\, V(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
$$

Since we only have to worry about the potential on the top of the box (and since we assume there is no charge inside the box), the Green's function solution may be written

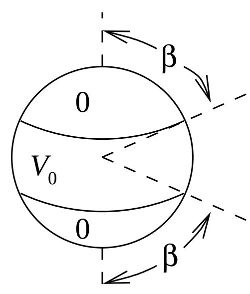
$$
\begin{aligned}
\Phi(\vec{x}\,) &= -\frac{1}{4\pi} \int_{z'=c} \Phi(\vec{x}\,') \frac{\partial G(\vec{x}, \vec{x}\,')}{\partial n'}\, da' \\
&= -\frac{1}{4\pi} \int_{z'=c} V(x', y') \frac{\partial G(\vec{x}, \vec{x}\,')}{\partial n'}\, dz'\, dy'
\end{aligned} \tag{3}
$$

Noting that the outward-pointing normal $\hat{n}'$ on the top of the box is in the $+\hat{z}'$ direction, we compute the normal derivative of (2)

$$
\begin{aligned}
\left.\frac{\partial G(\vec{x},\vec{x}\,')}{\partial n'}\right|_{z'=c} &= \left.\frac{\partial G(\vec{x},\vec{x}\,')}{\partial z'}\right|_{z'=c} \\
&= \frac{16\pi}{ab}\sum_{m,n}\frac{1}{\gamma_{mn}\sinh\gamma_{mn}c}\sin\frac{m\pi x}{a}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b} \\
&\qquad \times \sinh\gamma_{mn}z\Big(-\gamma_{mn}\cosh\gamma_{mn}[(c-z')]\Big)\Big|_{z'=c} \\
&= -4\pi\sum_{m,n}\frac{4}{ab\sinh\gamma_{mn}c}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sinh\gamma_{mn}z \\
&\qquad \times \sin\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b}
\end{aligned}
$$

Inserting this into (3) then straightforwardly gives the desired result. Note that the primes may be dropped from the double integral A_{mn} once it has been isolated from the rest of the expression.

2. The potential on the surface of a sphere of radius a is specified by



$$
V(\theta,\phi)=\begin{cases}0, & 0\le\theta<\beta\\ V_0, & \beta\le\theta\le\pi-\beta\\ 0, & \pi-\beta<\theta\le\pi\end{cases}
$$

There are no other charges in this problem.

a) Show that the potential outside the sphere may be expressed as

$$
\Phi(r,\theta,\phi)=\sum_{l=0,2,4,6,\ldots}V_0[P_{l+1}(\cos\beta)-P_{l-1}(\cos\beta)]\left(\frac{a}{r}\right)^{l+1}P_l(\cos\theta)
$$

where we take $P_{-1}(x)=0$. Note that Legendre polynomials satisfy the relation $(2l+1)P_l(x)=P'_{l+1}(x)-P'_{l-1}(x)$.

There are several ways of solving this problem. Perhaps the most straightforward is to realize from azimuthal symmetry that the potential necessarily admits a Legendre expansion

$$
\Phi(\vec{x}\,)=\sum_l \alpha_l\left(\frac{a}{r}\right)^{l+1}P_l(\cos\theta)
$$

The boundary conditions at $r = a$ gives

$$V(\theta) = \sum_l \alpha_l P_l(\cos\theta)$$

This is clearly a Legendre expansion for $V(\theta)$. The legendre orthogonality relation allows us to write the expansion coefficients α_l as

$$\alpha_l = \frac{2l+1}{2} \int_{-1}^{1} V(\cos\theta)\, P_l(\cos\theta)\, d(\cos\theta)$$

For the specified potential, this becomes

$$\alpha_l = \frac{2l+1}{2} \int_{-\cos\beta}^{\cos\beta} V_0 P_l(\cos\theta)\, d(\cos\theta) = \frac{V_0}{2} \int_{-\cos\beta}^{\cos\beta} (2l+1) P_l(x)\, dx$$

Using the Legendre relation given in the problem, we obtain

$$\begin{aligned}
\alpha_l &= \frac{V_0}{2} \int_{-\cos\beta}^{\cos\beta} [P'_{l+1}(x) - P'_{l-1}(x)]\, dx \\
&= \frac{V_0}{2} \Big[P_{l+1}(\cos\beta) - P_{l-1}(\cos\beta) - P_{l+1}(-\cos\beta) + P_{l-1}(-\cos\beta) \Big] \\
&= \frac{V_0}{2} (1 + (-)^l) [P_{l+1}(\cos\beta) - P_{l-1}(\cos\beta)]
\end{aligned}$$

This vanishes unless l is even (which should be obvious from the $z \to -z$ symmetry of the problem). The result is then

$$\Phi(r,\theta) = \sum_{l\,\text{even}} V_0 [P_{l+1}(\cos\beta) - P_{l-1}(\cos\beta)] \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$$

Note that $l = 0$ is allowed, and gives the monopole contribution. (An earlier version of this practice midterm excluded $l = 0$ from the sum, and this was a mistake.)

This problem could also have been solved by using the Dirichlet Green's function outside a sphere

$$G(\vec{x}, \vec{x}\,') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \frac{1}{r_>^{l+1}} Y^*_{lm}(\Omega') Y_{lm}(\Omega)$$

After slight manipulation, we would have ended up with a similar Legendre polynomial integration.

b) For fixed V_0, what angle β maximizes the quadrupole moment?

The quadrupole is given by $l = 2$. Hence the quadrupole moment is related to the α_2 term in the expansion. Since we do not care about normalization (we only care to maximize the moment) it is sufficient to write

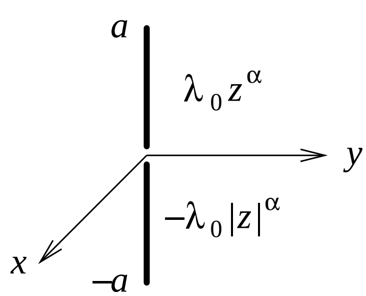
$$q_{2,0} \sim \alpha_2 \sim P_3(\cos\beta) - P_1(\cos\beta)$$

We extremize this quantity by taking a derivative with respect to β and setting the result to zero. Noting that taking a derivative simply 'undoes' the integration, we end up solving

$$\frac{dq_{2,0}}{d\beta} \sim P_2(\cos\beta)\sin\beta = \tfrac{1}{2}(3\cos^2\beta - 1)\sin\beta = 0$$

This is solved by $\beta = 0$ and $\beta = \cos^{-1}(\pm 1/\sqrt{3})$. Clearly the quadrupole moment vanishes when $\beta = 0$ (as the entire sphere is at a constant zero potential). Hence the maximum is when $\beta = \cos^{-1}(1/\sqrt{3})$, provided we restrict $0 \le \beta \le \pi/2$. Note that $\beta > \pi/2$ does not really make sense, except in a formal manner where $V_0 \to -V_0$ (corresponding to interchanging the integration limits). Technically, the problem should have asked to maximize the *magnitude* of the quadrupole moment instead of to make $q_{2,0}$ as positive as possible (which would depend on the sign of V_0).

3. A line charge on the z axis extends from $z = -a$ to $z = +a$ and has linear charge density varying as



$$\lambda(z) = \begin{cases} \lambda_0 z^\alpha, & 0 < z \le a \\ -\lambda_0 |z|^\alpha, & -a \le z < 0 \end{cases}$$

where α is a positive constant. The total charge on the $0 < z \le a$ segment is Q (and the charge on the $-a \le z < 0$ segment is $-Q$).

a) Calculate all of the multipole moments of the charge distribution. Make sure to indicate which moments are non-vanishing.

Noting that a uniformly charged line charge on the $+z$ axis has charge density

$$\rho = \frac{\lambda_0}{2\pi r^2}\delta(cos\theta - 1)$$

we see that a varying line charge yields

$$\rho = \frac{\lambda(r)}{2\pi r^2}\delta(\cos\theta - 1)$$

This may be checked by observing

$$dq = \rho\, d^3x = \rho\, r^2\, dr\, d\phi\, d(\cos\theta) = \lambda(r) dr \frac{d\phi}{2\pi}\bigg|_{\cos\theta=1}$$

(Note that there is no distinction between r and z for $\cos\theta = 1$.) Hence, for the positive and negative line charge, we have

$$\rho = \frac{\lambda_0 r^\alpha}{2\pi r^2}[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]$$

We may normalize λ_0 by integrating from 0 to a to obtain the total charge Q

$$Q = \int_0^a \lambda(z)\, dz = \int_0^a \lambda_0 z^\alpha\, dz = \frac{\lambda_0 a^{\alpha+1}}{\alpha+1}$$

Hence

$$\rho = Q\frac{\alpha+1}{2\pi a r^2}\left(\frac{r}{a}\right)^\alpha [\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]$$

The multipole moments are then

$$\begin{aligned} q_{lm} &= \int r^l Y^*_{lm}(\Omega) \rho\, r^2\, dr\, d\Omega \\ &= Q\frac{\alpha+1}{a}\int_0^a r^l \left(\frac{r}{a}\right)^\alpha dr \int Y^*_{lm}(\Omega)[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]\frac{d\Omega}{2\pi} \end{aligned}$$

By azimuthal symmetry, only the $m = 0$ moments are non-vanishing

$$\begin{aligned} q_{l,0} &= Q\frac{\alpha+1}{a}\frac{a^{l+1}}{\alpha+l+1}\int_{-1}^1 \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)] d(\cos\theta) \\ &= Qa^l\frac{\alpha+1}{\alpha+l+1}\sqrt{\frac{2l+1}{4\pi}}[P_l(1) - P_l(-1)] \\ &= Qa^l\frac{\alpha+1}{\alpha+l+1}\sqrt{\frac{2l+1}{4\pi}}[1 - (-)^l] \end{aligned}$$

Hence all moments vanish unless l is odd and m is zero. Then

$$q_{l,0} = Qa^l\frac{\alpha+1}{\alpha+l+1}\sqrt{\frac{2l+1}{\pi}} \qquad l \text{ odd}$$

b) Write down the multipole expansion for the potential in explicit form up to the first two non-vanishing terms.

The multipole expansion gives

$$
\begin{aligned}
\Phi &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\
&= \frac{1}{4\pi\epsilon_0}\sum_{l\,\mathrm{odd}}\sqrt{\frac{4\pi}{2l+1}}q_{l,0}\frac{P_l(\cos\theta)}{r^{l+1}} \\
&= \frac{2Q}{4\pi\epsilon_0 r}\sum_{l\,\mathrm{odd}}\frac{\alpha+1}{\alpha+l+1}\left(\frac{a}{r}\right)^l P_l(\cos\theta) \qquad (4)\\
&= \frac{2Q}{4\pi\epsilon_0}\left(\frac{(\alpha+1)a}{\alpha+2}\frac{P_l(\cos\theta)}{r^2}+\frac{(\alpha+1)a^3}{\alpha+4}\frac{P_3(\cos\theta)}{r^4}+\cdots\right) \\
&= \frac{2Q}{4\pi\epsilon_0}\left(\frac{(\alpha+1)a}{\alpha+2}\frac{\cos\theta}{r^2}+\frac{(\alpha+1)a^3}{2(\alpha+4)}\frac{5\cos^3\theta-3\cos\theta}{r^4}+\cdots\right)
\end{aligned}
$$

c) What is the dipole moment $\vec{p}$ in terms of Q, a and α?

Note that the dipole term in (4) has the form

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{2Q(\alpha+1)a}{\alpha+2}\frac{z}{r^3}$$

Comparing this with the dipole expression

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

gives

$$\vec{p} = \frac{2Q(\alpha+1)a}{\alpha+2}\hat{z}$$

Alternatively, one could compute directly

$$\vec{p} = \int \vec{x}\rho\, d^3x = \int_{-a}^{a}\vec{x}Q(\alpha+1)\frac{|z|^\alpha}{a^{\alpha+1}}\mathrm{sgn}(z)\,dz\bigg|_{x=y=0}$$

The z component is the only non-vanishing component

$$p_z = 2Q(\alpha+1)\int_0^a\frac{|z|^{\alpha+1}}{a^{\alpha+1}}\,dz = \frac{2Q(\alpha+1)a}{\alpha+2}$$