Physics 505 **Fall 2007**

Practice Final — Solutions

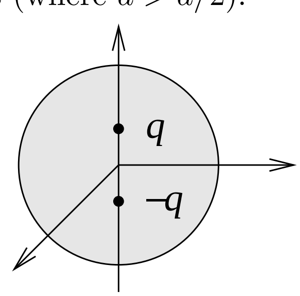
This final will be a three hour open book, open notes exam. Do all four problems.

1. Two point charges q and $-q$ are located on the z axis at $z = +d/2$ and $z = -d/2$, respectively.

   a) If the charges are isolated in space, show that the potential admits a Legendre expansion
   $$\Phi(r,\theta) = \frac{2q}{4\pi\epsilon_0}\sum_{l\ \mathrm{odd}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$
   where $r_< = \min(r, d/2)$ and $r_> = \max(r, d/2)$.

   From basic considerations, the Coulomb potential of the two point charges can be written as
   $$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-(d/2)\hat{z}|} - \frac{q}{|\vec{x}+(d/2)\hat{z}|}\right]$$
   We now recall the expansion
   $$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$
   where γ is the angle between $\vec{x}$ and $\vec{x}\,'$. Since the two charges are located on the z-axis, the angle γ is simply θ for the positive charge and $\pi - \theta$ for the negative charge. Thus
   $$\Phi = \frac{q}{4\pi\epsilon_0}\sum_l \frac{r_<^l}{r_>^{l+1}}[P_l(\cos\theta) - P_l(-\cos\theta)]$$
   Since $P_l(-\zeta) = (-1)^l P_l(\zeta)$, the even l components cancel out, and we are left with
   $$\Phi = \frac{2q}{4\pi\epsilon_0}\sum_{l\ \mathrm{odd}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

   b) Now consider the charges to be contained inside a linear dielectric sphere of permittivity ϵ and radius a (where $a > d/2$).

   Find the electric potential everywhere as an expansion in Legendre polynomials.

The dielectric sphere separates all of space into two regions, $r < a$ and $r > a$. We thus write down expansions for the electrostatic potential inside and outside the sphere, and match at the boundary. In particular, introduce

$$\Phi_{\text{in}} = \frac{2q}{4\pi\epsilon} \sum_{l \text{ odd}} \left(\frac{r_<^l}{r_>^{l+1}} + A_l r^l \right) P_l(\cos\theta)$$
$$\Phi_{\text{out}} = \frac{2q}{4\pi\epsilon_0} \sum_{l \text{ odd}} \left(\frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

where $r_< = \min(r, d/2)$ and $r_> = \max(r, d/2)$. Since we match at the surface $r = a$ we use the fact that $r_< = d/2$ and $r_> = r$ near this surface. The matching conditions are then on $D^\perp$ and $E^\parallel$. We find

$$D^\perp: \qquad -(l+1)\frac{(d/2)^l}{a^{l+2}} + lA_l a^{l-1} = -(l+1)\frac{B_l}{a^{l+2}}$$
$$E^\parallel: \qquad \frac{(d/2)^l}{a^{l+2}} + A_l a^{l-1} = \frac{\epsilon}{\epsilon_0}\frac{B_l}{a^{l+2}}$$

or, in matrix form

$$\begin{pmatrix} la^{2l+1} & l+1 \\ -a^{2l+1} & \epsilon/\epsilon_0 \end{pmatrix} \begin{pmatrix} A_l \\ B_l \end{pmatrix} = (d/2)^l \begin{pmatrix} l+1 \\ 1 \end{pmatrix}$$

which may be solved to yield

$$A_l = \frac{(\epsilon_r - 1)(l+1)}{(\epsilon_r + 1)l + 1} \frac{(d/2)^l}{a^{2l+1}}, \qquad B_l = \frac{2l+1}{(\epsilon_r+1)l+1}(d/2)^l$$

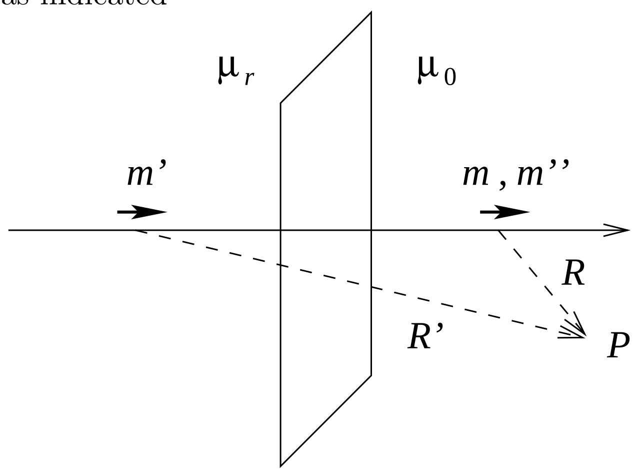
where $\epsilon_r = \epsilon/\epsilon_0$. This gives explicitly

$$\Phi_{\text{in}} = \frac{2q}{4\pi\epsilon} \sum_{l \text{ odd}} \left(\frac{r_<^l}{r_>^{l+1}} + \frac{(\epsilon_r - 1)(l+1)}{(\epsilon_r+1)l+1} \frac{(d/2)^l r^l}{a^{2l+1}} \right) P_l(\cos\theta)$$
$$\Phi_{\text{out}} = \frac{2q}{4\pi\epsilon_0} \sum_{l \text{ odd}} \left(\frac{2l+1}{(\epsilon_r+1)l+1} \frac{(d/2)^l}{r^{l+1}} \right) P_l(\cos\theta)$$

2. A point magnetic dipole $\vec{m}$ is located in vacuum pointing away from and a distance d away from a semi-infinite slab of material with relative permeability μ_r.

   a) Find the magnetic induction everywhere.

   Perhaps the most straightforward method to solve this problem is to use an image magnetic dipole. For a dipole $\vec{m}$ on the right of the slab, we introduce images $\vec{m}\,'$ and $\vec{m}\,''$ as indicated



   Following exercise 5.17 (with dipole sources instead of current sources), we find the strengths of the images to be

$$\vec{m}\,' = \frac{\mu_r - 1}{\mu_r + 1}\vec{m}, \qquad \vec{m}\,'' = \frac{2\mu_r}{\mu_r + 1}\vec{m}$$

   Since an isolated magnetic dipole has magnetic induction

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{R}(\hat{R}\cdot\vec{m}) - \vec{m}}{R^3}$$

   the effect of the images is to give a magnetic induction

$$\text{right:} \qquad \vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{R}(\hat{R}\cdot\vec{m}) - \vec{m}}{R^3} + \frac{\mu_r - 1}{\mu_r + 1}\frac{3\hat{R}'(\hat{R}'\cdot\vec{m}) - \vec{m}}{R'^3}\right]$$

$$\text{left:} \qquad \vec{B} = \frac{\mu_0}{4\pi}\frac{2\mu_r}{\mu_r + 1}\frac{3\hat{R}(\hat{R}\cdot\vec{m}) - \vec{m}}{R^3}$$

   where $\vec{R}$ and $\vec{R}'$ are as indicated in the figure. Introducing a rectangular coordinate system, we locate the dipole of strength $\vec{m} = (0, 0, m)$ at $(0, 0, d)$. Then $\vec{R} = (x, y, z - d)$ and $\vec{R}' = (x, y, z + d)$. Hence we have explicitly

$$\begin{aligned} \text{right:} \qquad \vec{B} = \frac{\mu_0 m}{4\pi}\bigg[&\frac{3(z-d)(x,y,z-d)}{(x^2+y^2+(z-d)^2)^{5/2}} - \frac{(0,0,1)}{(x^2+y^2+(z-d)^2)^{3/2}} \\ &+ \frac{\mu_r - 1}{\mu_r + 1}\left(\frac{3(z+d)(x,y,z+d)}{(x^2+y^2+(z+d)^2)^{5/2}} - \frac{(0,0,1)}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\bigg] \end{aligned}$$

$$\text{left:} \qquad \vec{B} = \frac{\mu_0 m}{4\pi}\frac{2\mu_r}{\mu_r + 1}\frac{3(z-d)(x,y,z-d)}{(x^2+y^2+(z-d)^2)^{5/2}} - \frac{(0,0,1)}{(x^2+y^2+(z-d)^2)^{3/2}}$$

b) What is the force on the dipole (magnitude and direction)?

The real dipole feels a force from its image. We may calculate

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}_{\text{image}})$$

where

$$\begin{aligned}\vec{m} \cdot \vec{B}_{\text{image}} &= \frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \left(\frac{3(z+d)^2}{(x^2 + y^2 + (z+d)^2)^{5/2}} - \frac{1}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right) \\ &= \frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{2(z+d)^2 - x^2 - y^2}{(x^2 + y^2 + (z+d)^2)^{5/2}}\end{aligned}$$

Instead of taking the complete gradient of this expression, we note from symmetry that the force can only act in the z direction. We thus compute the z derivative only

$$\begin{aligned}F_z &= \partial_z(\vec{m} \cdot \vec{B}_{\text{image}}) \\ &= \frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \left(\frac{4(z+d)}{(x^2 + y^2 + (z+d)^2)^{5/2}} - \frac{5(z+d)(2(z+d)^2 - x^2 - y^2)}{(x^2 + y^2 + (z+d)^2)^{7/2}} \right) \\ &= \frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{(z+d)(9x^2 + 9y^2 - 6(z+d)^2)}{(x^2 + y^2 + (z+d)^2)^{7/2}}\end{aligned}$$

This needs to be evaluated at the actual location of the dipole, $(x, y, z) = (0, 0, d)$. We thus obtain

$$F_z = \frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{(-6)}{(2d)^4} = -\frac{\mu_0 m^2}{4\pi} \frac{\mu_r - 1}{\mu_r + 1} \frac{3}{8d^4}$$

The dipole is hence attracted to the semi-infinite slab.

3. An infinitely long solenoid of radius a has N tightly wound turns per unit length. For a constant current I, elementary considerations tells us that the magnetic induction is uniform inside the solenoid. In cylindrical coordinates, $\vec{B} = \mu_0 N I \hat{z} \Theta(a - \rho)$ where $\Theta(\xi) = 1$ if $\xi > 0$ (and 0 otherwise) is the unit step function.

   This problem, however, involves a sinusoidal current $I(t) = I_0 e^{-i\omega t}$. In the following, consider only the inside of the solenoid and assume all fields vanish outside.

   a) By symmetry considerations, the time-dependent magnetic induction only has a non-vanishing z component, $B_z(\rho) e^{-i\omega t}$. Show that the electric field only has a component along the $\hat{\phi}$ direction. Consider the inside of the solenoid only.

   Note that the harmonic versions of Faraday's law and Ampère's law are

$$\vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0, \qquad \vec{\nabla} \times \vec{B} + \frac{i\omega}{c^2} \vec{E} = 0$$

We may solve Ampère's law for the electric field

$$\vec{E} = \frac{ic^2}{\omega}\vec{\nabla} \times \vec{B}$$

Using $\vec{B} = B_z(\rho)\hat{z}$ and the cylindrical coordinates expression for the curl, we obtain

$$\vec{E} = -\frac{ic^2}{\omega}B_z'\hat{\phi}$$

where $'$ denotes $d/d\rho$. This indicates that $\vec{E} = E_\phi(\rho)\hat{\phi}$ where $E_\phi = -(ic^2/\omega)B_z'$.

b) Find the exact solution for $B_z(\rho)$ inside the solenoid. Give your result in terms of the maximum current I flowing through the wires of the solenoid.

We now substitute the electric field into Faraday's law to obtain a second order differential equation in cylindrical coordinates

$$0 = \vec{\nabla} \times \vec{E} - i\omega\vec{B} = -\frac{ic^2}{\omega}\vec{\nabla} \times (B_z'\hat{\phi}) - i\omega B_z\hat{z} = -\frac{ic^2}{\omega}\frac{1}{\rho}\partial_\rho(\rho B_z')\hat{z} - i\omega B_z\hat{z}$$

The result is

$$\left(\frac{1}{\rho}\partial_\rho\rho\partial_\rho + \frac{\omega^2}{c^2}\right)B_z = 0$$

or

$$B_z'' + \frac{1}{\rho}B_z' + \frac{\omega^2}{c^2}B_z = 0$$

This is Bessel's equation, and has a solution

$$B_z(\rho) = AJ_0(\omega\rho/c)$$

(we do not use $N_0(\zeta)$ since the solution should not blow up at $\rho = 0$.) To obtain the constant A, we note that at the surface of the solenoid ($\rho = a$) the $H^{\|}$ matching conditions state

$$H^{\|} = K$$

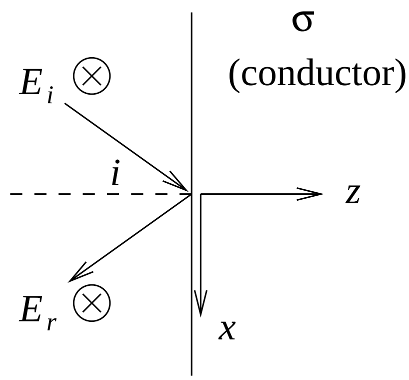
where K is the surface current density. Using $K = NI$, we obtain

$$\frac{1}{\mu_0}AJ_0\left(\frac{\omega a}{c}\right) = NI$$

or $A = \mu_0 NI/J_0(\omega a/c)$. Thus

$$B_z(\rho) = \mu_0 NI\frac{J_0(\omega\rho/c)}{J_0(\omega a/c)}$$

4. A plane polarized electromagnetic wave of frequency ω in free space is incident with angle i on the flat surface of an excellent conductor ($\mu = \mu_0$, $\epsilon = \epsilon_0$ and $\sigma \gg \omega\epsilon_0$) which fills the region $z > 0$.



Consider *only* linear polarization perpendicular to the plane of incidence.

a) If the incident wave is given by $\vec{E} = \vec{E}_i e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, show that (in the limit $\sigma \gg \omega\epsilon_0$) the magnitude of the electric field inside the conductor is

$$E_c = E_i \gamma \cos i \, e^{-z/\delta} e^{i(kx\sin i + z/\delta - \omega t)}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \qquad \text{and} \qquad \gamma = (1-i)\sqrt{\frac{2\epsilon_0\omega}{\sigma}}$$

The z direction is perpendicular to the flat surface of the conductor, while the x direction is parallel to it.

We use a complex dielectric constant

$$\epsilon' = \epsilon_0 + \frac{i\sigma}{\omega} \approx \frac{i\sigma}{\omega}$$

This gives

$$n' = \sqrt{\frac{\epsilon'}{\epsilon_0}} \approx \sqrt{\frac{i\sigma}{\epsilon_0\omega}} = (1+i)\sqrt{\frac{\sigma}{2\epsilon_0\omega}} = \frac{2}{\gamma}$$

as well as

$$k = \frac{\omega}{c}, \qquad k' = \frac{\omega n'}{c} = (1+i)\sqrt{\frac{\omega\mu_0\sigma}{2}} = \frac{(1+i)}{\delta}$$

For $\vec{E}$ perpendicular to the plane of incidence, the 'transmitted' wave is given by

$$E' = E_i \frac{2n\cos i}{n\cos i + \sqrt{n'^2 - n^2\sin^2 i}}$$

Hence

$$\begin{aligned} E_c = E' e^{i(\vec{k}'\cdot\vec{x}-\omega t)} &= E_i \frac{2n\cos i}{n\cos i + \sqrt{n'^2 - n^2\sin^2 i}} e^{i(k'z\cos r + k'x\sin r - \omega t)} \\ &= E_i \frac{2\cos i}{\cos i + \sqrt{n'^2 - \sin^2 i}} e^{i((1+i)(z/\delta)\cos r + kxn'\sin r - \omega t)} \end{aligned}$$

However, in the limit $|n'| \gg 1$ we find

$$n' \cos r = \sqrt{n'^2 - \sin^2 i} \approx n'$$

so that $\cos r \approx 1$. Using these approximations in the above, we arrive at

$$E_c = E_i \frac{2\cos i}{n} e^{i((1+i)z/\delta + kx\sin i - \omega t)} = E_i \gamma \cos i \, e^{-z/\delta} e^{i(kx\sin i + z/\delta - \omega t)}$$

b) Show that the time averaged power per unit area flowing into the conductor is given by $S^\perp = \epsilon_0 |E_i|^2 \omega\delta \cos^2 i$.

There are numerous was of computing the power flowing into the conductor. We can compute the incident minus the reflected power, or we may simply compute the transmitted power just within the conductor. We could also compute the total power dissipated in the conductor by integrating through the entire depth of the conductor. By energy conservation, all these should give the same result.

Suppose we compute the power dissipated in the conductor. By Ohm's law, we write

$$P = \frac{1}{2}\vec{E} \cdot \vec{J}^* = \frac{\sigma}{2}|\vec{E}\,|^2$$

Using the above expression, this gives

$$P = \frac{\sigma}{2}|E_i|^2|\gamma|^2 \cos^2 i \, e^{-2z/\delta} = 2\epsilon_0\omega|E_i|^2 \cos^2 i \, e^{-2z/\delta}$$

Note that this is the power density (per volume) lost in the conductor. To get the power lost per unit cross-sectional area, we have to integrate this along the z direction

$$P^\perp = \int_0^\infty P\, dz = \epsilon_0 |E_i|^2 \omega\delta \cos^2 i$$

By conservation of energy, this must be equal to the power flowing into the conductor.