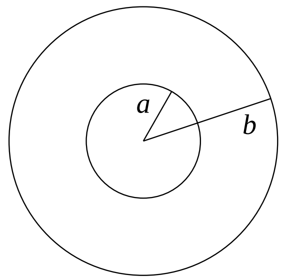
Physics 505 **Fall 2007**

Midterm — Solutions

This midterm is a two hour open book, open notes exam. Do all three problems.

[30 pts] 1. Consider a two-dimensional problem defined in the region between concentric circles of radii a and b.



[10] *a*) Using polar coordinates, the Dirichlet Green's function may be expanded as

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{\infty} g_m(\rho,\rho')e^{im(\phi-\phi')}$$

Write down the appropriate differential equation for $g_m(\rho,\rho')$.

In two dimensions, the Green's function satisfies

$$\nabla'^2 G(\vec{x},\vec{x}\,') = -4\pi\delta^{(2)}(\vec{x}-\vec{x}\,')$$

Using polar coordinates, we note that

$$\nabla'^2 = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} + \frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}$$

and

$$\delta^{(2)}(\vec{x}-\vec{x}\,') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

As a result, we have

$$\begin{aligned}\left(\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} + \frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}\right)G(\rho,\phi;\rho',\phi') &= -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\\ &= -\sum_m \frac{2}{\rho}\delta(\rho-\rho')e^{im(\phi-\phi')}\end{aligned}$$

where we have used the completeness relation

$$\sum_m e^{im(\phi-\phi')} = 2\pi\delta(\phi-\phi')$$

Inserting the expansion

$$G(\rho,\phi;\rho',\phi') = \sum_m g_m(\rho,\rho') e^{im(\phi-\phi')}$$

into the above and matching powers of $e^{i(\phi-\phi')}$ then gives the differential equation

$$\left(\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{m^2}{\rho'^2}\right) g_m(\rho,\rho') = -\frac{2}{\rho}\delta(\rho-\rho') \tag{1}$$

[20] *b*) Solve the Green's function equation for $g_m(\rho,\rho')$ subject to Dirichlet boundary conditions and write down the result for $G(\rho,\phi;\rho',\phi')$. Note that the $m=0$ case may need to be treated separately.

We start with the $m \neq 0$ case. The homogeneous equation corresponding to the Green's function equation (1) is

$$\left(\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{m^2}{\rho'^2}\right) g_m(\rho,\rho') = 0$$

This is easy to solve as it is equidimensional in ρ'. The two independent solutions are of the form ρ'^m and ρ'^{-m}. Because of the delta-function source in (1), we break up the ρ' interval into $a \le \rho' \le \rho$ and $\rho \le \rho' \le b$. Hence we write

$$g_m(\rho,\rho') = \begin{cases} Au(\rho') & a \le \rho' \le \rho \\ Bv(\rho') & \rho \le \rho' \le b \end{cases}$$

where

$$u(\rho') = \left(\frac{\rho'}{a}\right)^m - \left(\frac{a}{\rho'}\right)^m, \qquad v(\rho') = \left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m \tag{2}$$

are appropriately chosen to satisfy the Dirichlet boundary conditions $g_m(\rho,a)=0$ and $g_m(\rho,b)=0$. Note that these expressions are valid for both positive and negative m. From (1), we must now satisfy the matching and jump conditions

$$g_< = g_>, \qquad \frac{\partial}{\partial\rho'}g_< = \frac{\partial}{\partial\rho'}g_> + \frac{2}{\rho} \tag{3}$$

where $g_<$ and $g_>$ are the values of $g_m(\rho,\rho')$ for ρ' immediately to the left and right of the delta function at ρ, respectively. These conditions give rise to a set of two equations which may be solved to determine the two unknowns A and B. Alternatively, by symmetry of the Green's function, we may write

$$g_m(\rho,\rho') = Au(\rho_<)v(\rho_>)$$

where $\rho_< = \min(\rho,\rho')$ and $\rho_> = \max(\rho,\rho')$, and where A is a ρ and ρ' independent constant. In this case, the first condition of (3) is automatically satisfied, while the second one gives

$$Au'(\rho)v(\rho) = Au(\rho)v'(\rho) + \frac{2}{\rho}$$

or equivalently

$$A = -\frac{2}{\rho} \begin{vmatrix} u(\rho) & v(\rho) \\ u'(\rho) & v'(\rho) \end{vmatrix}^{-1}$$

Note that the determinant is simply the Wronskian of $u(\rho)$ and $v(\rho)$. In any case, using (2), we see that

$$\begin{aligned} \begin{vmatrix} u(\rho) & v(\rho) \\ u'(\rho) & v'(\rho) \end{vmatrix} &= \frac{m}{\rho}\Bigg[\left(\left(\frac{\rho}{a}\right)^m - \left(\frac{a}{\rho}\right)^m\right)\left(\left(\frac{\rho}{b}\right)^m + \left(\frac{b}{\rho}\right)^m\right) \\ &\qquad - \left(\left(\frac{\rho}{a}\right)^m + \left(\frac{a}{\rho}\right)^m\right)\left(\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right)\Bigg] \\ &= \frac{2m}{\rho}\left[\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m\right] \end{aligned}$$

This gives

$$A = -\frac{1}{m}\left[\left(\frac{b}{a}\right)^m - \left(\frac{a}{b}\right)^m\right]^{-1}$$

so that

$$g_m(\rho,\rho') = -\frac{u(\rho_<)v(\rho_>)}{m[(b/a)^m - (a/b)^m]} \qquad (m \neq 0) \tag{4}$$

where $u(\rho)$ and $v(\rho)$ are given in (2).

When $m = 0$, the Green's function equation (1) reduces to

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'}g_0(\rho,\rho') = -\frac{2}{\rho}\delta(\rho-\rho')$$

In this case, the two linearly independent solutions to the homogeneous equation are 1 (ie a constant) and $\log\rho'$. The Dirichlet boundary conditions are then satisfied with

$$u(\rho') = \log\left(\frac{\rho'}{a}\right), \qquad v(\rho') = \log\left(\frac{\rho'}{b}\right)$$

This time, the Wronskian is

$$\begin{vmatrix} u(\rho) & v(\rho) \\ u'(\rho) & v'(\rho) \end{vmatrix} = \frac{1}{\rho}\left[\log\left(\frac{\rho}{a}\right) - \log\left(\frac{\rho}{b}\right)\right] = \frac{1}{\rho}\log\left(\frac{b}{a}\right)$$

so that

$$A = -\frac{2}{\rho}\begin{vmatrix} u(\rho) & v(\rho) \\ u'(\rho) & v'(\rho) \end{vmatrix}^{-1} = -\frac{2}{\log(b/a)}$$

and

$$g_0(\rho,\rho') = -\frac{2\log(\rho_</a)\log(\rho_>/b)}{\log(b/a)} \tag{5}$$

Finally, combining (4) and (5) gives the complete Green's function

$$G(\rho,\phi;\rho',\phi') = -\frac{2\log(\rho_</a)\log(\rho_>/b)}{\log(b/a)} - \sum_{m\neq 0}\frac{[(\rho_</a)^m-(a/\rho_<)^m][(\rho_>/b)^m-(b/\rho_>)^m]}{m[(b/a)^m-(a/b)^m]}e^{im(\phi-\phi')}$$

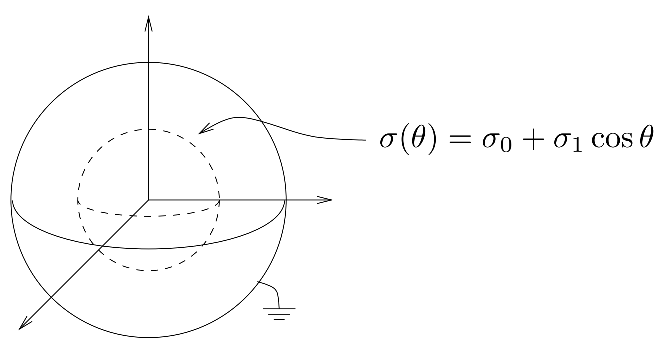
Since the prefactor to $e^{im(\phi-\phi')}$ is even under the replacement $m\to -m$, the Green's function may equivalently be written as

$$G(\rho,\phi;\rho',\phi') = \frac{2\log(\rho_</a)\log(b/\rho_>)}{\log(b/a)} + \sum_{m=1}^{\infty}\frac{2[(\rho_</a)^m-(a/\rho_<)^m][(b/\rho_>)^m-(\rho_>/b)^m]}{m[(b/a)^m-(a/b)^m]}\cos[m(\phi-\phi')]$$

or

$$G(\rho,\phi;\rho',\phi') = \frac{2\log(\rho_</a)\log(b/\rho_>)}{\log(b/a)} + \sum_{m=1}^{\infty}\frac{2}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\frac{[1-(a/\rho_<)^{2m}][1-(\rho_>/b)^{2m}]}{[1-(a/b)^{2m}]}\cos[m(\phi-\phi')]$$

[35 pts] 2. A spherical surface of radius a and surface-charge density $\sigma(\theta)=\sigma_0+\sigma_1\cos\theta$ is placed concentrically inside a grounded conducting sphere of radius b. Here θ is the standard polar angle in spherical coordinates.



[20] *a*) Find the potential $\Phi(r,\theta,\phi)$ everywhere inside the conducting sphere.

Since this problem focuses on the interior of a conducting sphere of radius b, we may use the Dirichlet Green's function

$$G(\vec{x},\vec{x}\,') = \sum_{l,m}\frac{4\pi}{2l+1}r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)Y_l^m(\Omega)Y_l^{m\,*}(\Omega') \qquad (6)$$

In general, the potential inside the conducting sphere is given by

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\int_V G(\vec{x},\vec{x}\,')\rho(\vec{x}\,')d^3x' - \frac{1}{4\pi}\int_S \Phi(\vec{x}\,')\frac{\partial G}{\partial n'}da'$$

However the surface term does not contribute since the potential $\Phi(\vec{x}\,')$ vanishes on the surface of the grounded conducting sphere. As a result, we are left to evaluate

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\int_V G(\vec{x},\vec{x}\,')\rho(\vec{x}\,')d^3x'$$

where

$$\rho(\vec{x}\,') = \sigma(\theta')\delta(r'-a) = [\sigma_0 + \sigma_1\cos\theta']\delta(r'-a)$$

Using the Green's function of (6) and using the $\delta(r'-a)$ to kill the r' integral gives

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)Y_l^m(\Omega)\int\sigma(\theta')Y_l^{m\,*}(\Omega')a^2d\Omega'$$

where $r_< = \min(r,a)$ and $r_> = \max(r,a)$. Since the charge distribution $\sigma(\theta')$ is azimuthally symmetric, only the $m=0$ terms survive in the sum, and we are left with a Legendre polynomial series

$$\Phi(r,\theta) = \frac{2\pi a^2}{4\pi\epsilon_0}\sum_l r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos\theta)\int_{-1}^1\sigma(\theta')P_l(\cos\theta')d(\cos\theta')$$

We now use

$$\sigma(\theta') = \sigma_0 P_0(\cos\theta') + \sigma_1 P_1(\cos\theta')$$

and the orthogonality of Legendre polynomials

$$\int_{-1}^1 P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{l,l'}$$

to obtain

$$\begin{aligned}\Phi(r,\theta) &= \frac{a^2}{\epsilon_0}\sum_l r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos\theta)\left[\sigma_0\delta_{l,0} + \frac{1}{3}\sigma_1\delta_{l,1}\right]\\ &= \frac{a^2}{\epsilon_0}\left[\sigma_0\left(\frac{1}{r_>} - \frac{1}{b}\right) + \frac{1}{3}\sigma_1 r_<\left(\frac{1}{r_>^2} - \frac{r_>}{b^3}\right)\cos\theta\right]\end{aligned}$$

Explicitly, this gives

$$\Phi(r,\theta) = \begin{cases}\dfrac{a^2}{\epsilon_0}\left[\sigma_0\left(\dfrac{1}{a} - \dfrac{1}{b}\right) + \dfrac{1}{3}\sigma_1 r\left(\dfrac{1}{a^2} - \dfrac{a}{b^3}\right)\cos\theta\right] & r < a\\ \dfrac{a^2}{\epsilon_0}\left[\sigma_0\left(\dfrac{1}{r} - \dfrac{1}{b}\right) + \dfrac{1}{3}\sigma_1 a\left(\dfrac{1}{r^2} - \dfrac{r}{b^3}\right)\cos\theta\right] & r > a\end{cases} \tag{7}$$

An alternate means of solving this problem is to solve Laplace's equation separately for $r < a$ and for $a < r < b$, and to match the two solutions at the location of the charged surface, $r = a$. Taking boundary conditions into account, we may write

$$\begin{aligned} \Phi_< &= \sum_l \alpha_l r^l P_l(\cos\theta) & (r < a) \\ \Phi_> &= \sum_l \beta_l \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) P_l(\cos\theta) & (a < r < b) \end{aligned} \tag{8}$$

where these forms have been chosen to satisfy the boundary conditions at $r = 0$ and $r = b$, respectively. The matching conditions at the surface $r = a$ are that the potential is continuous, $\Phi_< = \Phi_>\big|_{r=a}$ and that the jump in the perpendicular component of the electric field is given by σ/ϵ_0, namely $E_r^> = E_r^< + \sigma/\epsilon_0\big|_{r=a}$ or $\partial\Phi_</\partial r = \partial\Phi_>/\partial r + \sigma/\epsilon_0\big|_{r=a}$. These two conditions lead to the simultaneous equations

$$\alpha_l a^{2l+1} - \beta_l\left(1 - (a/b)^{2l+1}\right) = 0$$
$$l\alpha_l a^{2l+1} + \beta_l\left((l+1) + l(a/b)^{2l+1}\right) = \sigma_l a^{l+2}/\epsilon_0$$

which may be written in matrix form

$$\begin{pmatrix} 1 & -1 + (a/b)^{2l+1} \\ l & l+1+l(a/b)^{2l+1} \end{pmatrix} \begin{pmatrix} \alpha_l a^{2l+1} \\ \beta_l \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_l a^{l+2}/\epsilon_0 \end{pmatrix}$$

This may be solved to give

$$\begin{aligned} \begin{pmatrix} \alpha_l a^{2l+1} \\ \beta_l \end{pmatrix} &= \frac{1}{2l+1} \begin{pmatrix} l+1+l(a/b)^{2l+1} & 1-(a/b)^{2l+1} \\ -l & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sigma_l a^{l+2}/\epsilon_0 \end{pmatrix} \\ &= \frac{\sigma_l a^{l+2}}{(2l+1)\epsilon_0} \begin{pmatrix} 1-(a/b)^{2l+1} \\ 1 \end{pmatrix} \end{aligned}$$

In particular

$$\alpha_0 = \frac{\sigma_0 a}{\epsilon_0}\left(1 - \frac{a}{b}\right), \qquad \alpha_1 = \frac{\sigma_1}{3\epsilon_0}\left(1 - \left(\frac{a}{b}\right)^3\right)$$

and

$$\beta_0 = \frac{\sigma_0 a^2}{\epsilon_0}, \qquad \beta_1 = \frac{\sigma_1 a^3}{3\epsilon_0}$$

Substituting these coefficients into (8) reproduces the potential (7) obtained above using the Green's function method.

[10] *b*) What is the induced surface-charge density on the interior surface of the conducting sphere?

The induced surface-charge density is given by

$$\sigma = -\epsilon_0 E_r\Big|_{r=b} = \epsilon_0 \frac{\partial\Phi}{\partial r}\Big|_{r=b}$$

Using the expression for $\Phi(r > a)$ obtained in (7), we see that

$$\begin{aligned}\sigma &= a^2 \frac{\partial}{\partial r}\left[\sigma_0\left(\frac{1}{r}-\frac{1}{b}\right)+\frac{1}{3}\sigma_1 a\left(\frac{1}{r^2}-\frac{r}{b}^3\right)\cos\theta\right]_{r=b} \\ &= -\left[\sigma_0\left(\frac{a}{b}\right)^2+\sigma_1\left(\frac{a}{b}\right)^3\cos\theta\right]\end{aligned} \tag{9}$$

[5] *c*) What is the total induced charge on the interior surface of the conducting sphere?

The total induced charge is obtained by integrating (9) over the area of the conducting sphere

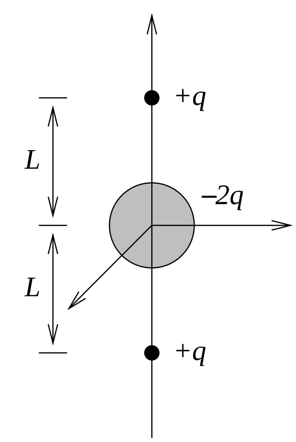
$$Q_{\text{induced}} = \int_{r=b} \sigma\, da = -\sigma_0\left(\frac{a}{b}\right)^2 (4\pi b^2) = -4\pi a^2\sigma_0$$

Note that the dipole term proportional to σ_1 integrates to zero over the entire surface of the sphere. This is just the negative of the total charge of the surface at $r = a$

$$q = (\text{average surface charge density}) \times (\text{area}) = \sigma_0(4\pi a^2)$$

Even without knowning the result of part b, this can be obtained directly by elementary application of Gauss' law inside a hollow conductor.

[35 pts] 3. A solid (ungrounded) conducting sphere of radius a and charge $-2q$ is located at the origin. A point charge of $+q$ is placed above the conducting sphere at a distance L from the origin, and another one (also of charge $+q$) is placed at a distance L below the origin.



[15] *a*) Find the potential $\Phi(\vec{x}\,)$ everywhere outside the conducting sphere. (Take $\Phi = 0$ at infinity.)

Perhaps the most straightforward way to approach this problem is to use the method of images. The image charge corresponding to the $+q$ charge located at a distance L from the center is $-q(a/L)$, and its location is a^2/L from the center. If the conducting sphere is grounded, the potential is then

$$\Phi_{\text{grounded}} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}-L\hat{z}|}+\frac{1}{|\vec{x}+L\hat{z}|}-\frac{a/L}{|\vec{x}-(a^2/L)\hat{z}|}-\frac{a/L}{|\vec{x}+(a^2/L)\hat{z}|}\right]$$

However, the conducting sphere is actually ungrounded, and has a total charge $-2q$ on it. Taking into account the two image changes, the effective charge on the sphere is $q_{\text{eff}} = -2q + 2q(a/L)$. Hence the potential is

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{-2+2a/L}{|\vec{x}\,|}+\frac{1}{|\vec{x}-L\hat{z}|}+\frac{1}{|\vec{x}+L\hat{z}|}-\frac{a/L}{|\vec{x}-(a^2/L)\hat{z}|}-\frac{a/L}{|\vec{x}+(a^2/L)\hat{z}|}\right] \tag{10}$$

[5] *b*) What is the potential of the conducting sphere?

The surface of the conducting sphere is given by $r = a$. Since the method of images guarantees that $\Phi_{\text{grounded}}(r=a) = 0$, and since we may rewrite (10) as

$$\Phi = \Phi_{\text{grounded}} - \frac{q}{2\pi\epsilon_0}\frac{1-a/L}{|\vec{x}\,|}$$

we immediately see that

$$\Phi(r=a) = -\frac{q}{2\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{L}\right)$$

This is the potential of the conducting sphere.

[15] *c*) Calculate the multipole moments q_{lm}. Make sure to indicate which moments are non-vanishing.

In order to calculate the multipole moments, we first rewrite (10) using the azimuthally symmetric expansion

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

where $\cos\gamma = \hat{x}\cdot\hat{x}'$, and where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. Since the charges are on the $\hat{z}$ axis, the angle γ is either θ or $\pi-\theta$. The expansion of (10) is then

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{-2+2a/L}{r}+\sum_l\left(\frac{r_<^l}{r_>^{l+1}}-\frac{a}{L}\frac{(a^2/L)^l}{r^{l+1}}\right)\big(P_l(\cos\theta)+P_l(-\cos\theta)\big)\right]$$

Since $P_l(-x) = (-1)^l P_l(x)$, this expression simplifies to

$$\begin{aligned}\Phi &= \frac{q}{2\pi\epsilon_0}\left[\frac{-1+a/L}{r}+\sum_{l=0,2,4,\ldots}\left(\frac{r_<^l}{r_>^{l+1}}-\frac{a^{2l+1}}{(rL)^{l+1}}\right)P_l(\cos\theta)\right]\\ &= \frac{q}{2\pi\epsilon_0}\left[\left(\frac{1}{r_>}-\frac{1}{r}\right)+\sum_{l=2,4,\ldots}\left(\frac{r_<^l}{r_>^{l+1}}-\frac{a^{2l+1}}{(rL)^{l+1}}\right)P_l(\cos\theta)\right]\end{aligned}$$

The asymptotic potential far away from the charges is obtained by taking $r_< = L$ and $r_> = r$. In this case

$$\begin{aligned}\Phi &= \frac{q}{2\pi\epsilon_0}\sum_{l=2,4,\ldots}\left(\frac{L^l}{r^{l+1}} - \frac{a^{2l+1}}{(rL)^{l+1}}\right)P_l(\cos\theta)\\ &= \frac{q}{2\pi\epsilon_0}\sum_{l=2,4,\ldots} L^l\left(1-\left(\frac{a}{L}\right)^{2l+1}\right)\frac{P_l(\cos\theta)}{r^{l+1}}\\ &= \frac{q}{2\pi\epsilon_0}\sum_{l=2,4,\ldots}\sqrt{\frac{4\pi}{2l+1}}L^l\left(1-\left(\frac{a}{L}\right)^{2l+1}\right)\frac{Y_l^0(\Omega)}{r^{l+1}}\end{aligned}$$

Comparing with with the multipole expansion

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_l^m(\Omega)}{r^{l+1}}$$

allows us to identify the multipole moments

$$q_{l,0} = \sqrt{\frac{2l+1}{\pi}}qL^l\left(1-\left(\frac{a}{L}\right)^{2l+1}\right) \qquad l = 2,4,6,\ldots \tag{11}$$

Alternatively, these moments may be obtained by realizing that this problem is equivalent to a five point charge problem by the method of images. The five point charges are the obvious ones leading to the potential (10)

$$\begin{aligned}\rho(\vec{x}) = q\Big[&-2(1-a/L)\delta^{(3)}(\vec{x}\,) + \delta^{(3)}(\vec{x}-L\hat{z}) + \delta^{(3)}(\vec{x}+L\hat{z})\\ &- (a/L)\delta^{(3)}(\vec{x}-(a^2/L)\hat{z}) - (a/L)\delta^{(3)}(\vec{x}+(a^2/L)\hat{z})\Big]\end{aligned}$$

In general, the multipole moments are defined by

$$q_{lm} = \int \rho(\vec{x}\,)r^l Y_l^{m\,*}(\Omega)d^3x$$

However, because of azimuthal symmetry, only the $m = 0$ moments are non-vanishing. In this case

$$\begin{aligned}q_{l,0} &= \sqrt{\frac{2l+1}{4\pi}}\int \rho(\vec{x}\,)r^l P_l(\cos\theta)d^3x\\ &= \sqrt{\frac{2l+1}{4\pi}}q\Big[-2\left(1-\frac{a}{L}\right)\delta_{l,0} + L^l P_l(1) + L^l P_l(-1)\\ &\qquad - \frac{a}{L}\left(\frac{a^2}{L}\right)^l P_l(1) - \frac{a}{L}\left(\frac{a^2}{L}\right)^l P_l(-1)\Big]\end{aligned}$$

Since $P_l(1) = 1$ and $P_l(-1) = (-1)^l$, this simplifies to

$$q_{l,0} = \sqrt{\frac{2l+1}{\pi}}q\left[-\left(1-\frac{a}{L}\right)\delta_{l,0} + L^l\left(1-\left(\frac{a}{L}\right)^{2l+1}\right)\right] \qquad l \text{ even}$$

Noting that the $l = 0$ term vanishes, we see that this result is identical to (11).