Physics 505 **Fall 2007**

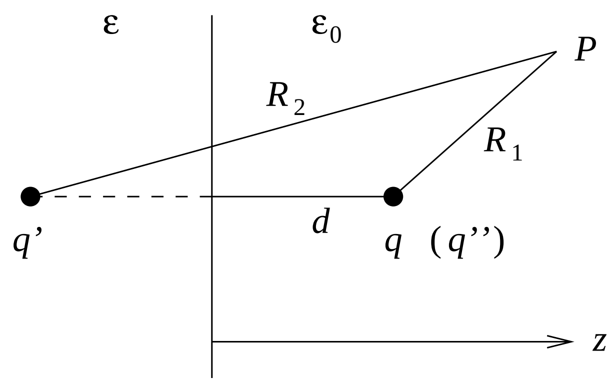
Final Exam — Solutions

This final is a three hour open book, open notes exam. Do all four problems.

[25 pts] 1. A point electric dipole with dipole moment $\vec{p}$ is located in vacuum pointing away from and a distance d away from the flat surface of a semi-infinite dielectric with permittivity ϵ.

[15] *a*) Find the electric potential Φ everywhere.

This problem can be solved by the method of images for a dielectric. Recall that for a point charge q located a distance d from the flat surface of a semi-infinite dielectric



the electric potential is given by

$$\Phi(z>0) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$$
$$\Phi(z<0) = \frac{1}{4\pi\epsilon}\frac{q''}{R_1}$$

where

$$q' = -q\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right), \qquad q'' = q\frac{2\epsilon}{\epsilon+\epsilon_0}$$

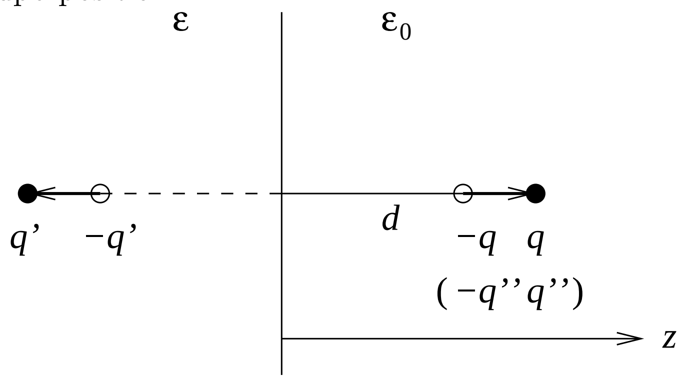
and where the position vectors are

$$\vec{R}_1 = (x, y, z-d), \qquad \vec{R}_2 = (x, y, z+d)$$

Here we have assume that the physical charge q is located at $(0, 0, d)$ and the observer is at point P given by (x, y, z). By substituting in q' and q'', the potential is more explicitly written as

$$\begin{aligned}\Phi(z>0) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{1}{R_2}\right)\\ \Phi(z<0) &= \frac{q}{4\pi\epsilon}\left(\frac{2\epsilon}{\epsilon+\epsilon_0}\frac{1}{R_1}\right)\end{aligned} \qquad (1)$$

Since a point electric dipole may be obtained by taking two charges $-q$ and $+q$ separated by a distance l in the limit $l \to 0$, the dipole problem may be solved by linear superposition



Since the electric potential for a dipole in free space is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

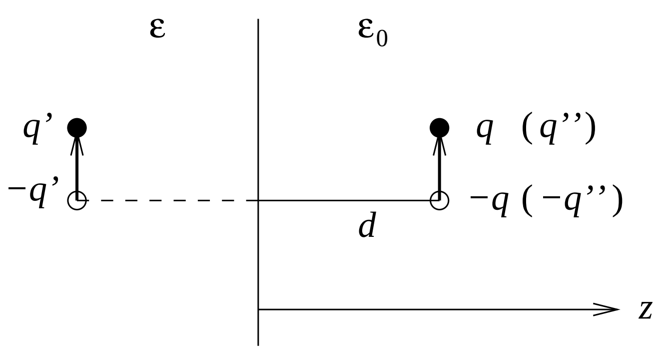
the electric dipole generalization of the point charge solution (1) is then

$$\begin{aligned}\Phi(z>0) &= \frac{\vec{p}}{4\pi\epsilon_0}\cdot\left(\frac{\vec{R}_1}{R_1^3}+\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{\vec{R}_2}{R_2^3}\right)\\ \Phi(z<0) &= \frac{\vec{p}}{4\pi\epsilon}\cdot\left(\frac{2\epsilon}{\epsilon+\epsilon_0}\frac{\vec{R}_1}{R_1^3}\right)\end{aligned}\tag{2}$$

where $\vec{p} = p\hat{z}$ is pointing away from the dielectric. Note that, *according to the figure*, the image dipole points in the opposite direction as the physical one, so long as we define the direction to be from $-q'$ to $+q'$. This is what accounts for the sign difference between the two terms in the first lines of (1) and (2). In reality, however, since the image charge q' has the opposite sign as q (assuming $\epsilon > \epsilon_0$), the image dipole actually points in the same direction as the physical one. This physical result is consistent with the plus sign in the first line of (2), which shows that both dipoles point in the same direction.

[10] *b*) What is the electric potential if the dipole is instead oriented parallel to the surface of the dielectric?

Note that the orientation of the image dipole is different for the parallel configuration

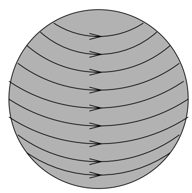


As a result, the potential is given instead by

$$\Phi(z>0) = \frac{\vec{p}}{4\pi\epsilon_0}\cdot\left(\frac{\vec{R}_1}{R_1^3} - \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{\vec{R}_2}{R_2^3}\right)$$
$$\Phi(z<0) = \frac{\vec{p}}{4\pi\epsilon}\cdot\left(\frac{2\epsilon}{\epsilon+\epsilon_0}\frac{\vec{R}_1}{R_1^3}\right)$$

where $\vec{p} = p\hat{x}$ is pointing parallel to the surface of the dielectric. Note that the image solution can be generalized for a dipole at an arbitrary angle relative to the surface of the dielectric.

[25 pts] 2. A wire coil is wound around the surface of a solid sphere of radius a and relative permeability μ_r. The coil is designed in such a way that it carries a surface current density $\vec{K} = \hat{\phi}(I/a)\sin\theta$.



Find the magnetic induction $\vec{B}$ everywhere.

Despite the presence of a surface current, this problem may be solved using a magnetic scalar potential approach. The trick is to realize that the two separate regions $r<a$ and $r>a$ are both current-free regions of space. This allows us to introduce 'inside' ($r<a$) and 'outside' ($r>a$) potentials

$$\vec{H}^{\text{in}} = -\vec{\nabla}\Phi_M^{\text{in}}, \qquad \vec{H}^{\text{out}} = -\vec{\nabla}\Phi_M^{\text{out}}$$

where Φ_M^{in} and Φ_M^{out} solve Laplace's equation, $\nabla^2\Phi_M = 0$. Using spherical coordinates, and taking azimuthal symmetry into account, we may write

$$\begin{aligned}\Phi_M^{\text{in}} &= \sum_l A_l r^l P_l(\cos\theta)\\ \Phi_M^{\text{out}} &= \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)\end{aligned} \tag{3}$$

The effect of the surface current $\vec{K} = \hat{\phi}(I/a)\sin\theta$ shows up in the matching conditions at $r=a$

$$\hat{r}\cdot(\vec{B}^{\text{out}} - \vec{B}^{\text{in}})|_{r=a} = 0, \qquad \hat{r}\times(\vec{H}^{\text{out}} - \vec{H}^{\text{in}})|_{r=a} = \vec{K}$$

In explicit components, these conditions are

$$B_r^{\text{in}} = B_r^{\text{out}}, \qquad H_\theta^{\text{in}} - H_\theta^{\text{out}} = \frac{I}{a}\sin\theta \qquad (\text{at } r=a)$$

Given (3), the appropriate components of the magnetic induction and magnetic field are

$$B_r^{\rm in} = -\mu \sum_l l A_l r^{l-1} P_l(\cos\theta)$$
$$B_r^{\rm out} = \mu_0 \sum_l (l+1) \frac{B_l}{r^{l+2}} P_l(\cos\theta)$$

and

$$H_\theta^{\rm in} = \sum_l A_l r^{l-1} P_l'(\cos\theta)\sin\theta$$
$$H_\theta^{\rm out} = \sum_l \frac{B_l}{r^{l+2}} P_l'(\cos\theta)\sin\theta$$

By orthogonality, the matching conditions must independently hold for each value of l. Noting that

$$\frac{I}{a}\sin\theta = \frac{I}{a}P_1'(\cos\theta)\sin\theta$$

we see that the matching conditions are

$$\begin{aligned}(l+1)B_l + \mu_r l A_l a^{2l+1} &= 0\\ B_l - A_l a^{2l+1} &= I a^{l+1}\delta_{l,1}\end{aligned}$$

These equations are homogeneous, except for $l = 1$. As a result, only the $l = 1$ mode contributes, with a solution

$$A_1 = -\frac{I}{a}\frac{1}{1+\mu_r/2}, \qquad B_1 = Ia^2\frac{\mu_r/2}{1+\mu_r/2}$$

The magnetic scalar potential is then

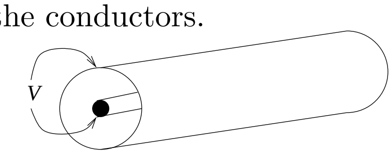
$$\begin{aligned}\Phi_M^{\rm in} &= -\frac{I}{a}\frac{1}{1+\mu_r/2} r\cos\theta = -\frac{I}{a}\frac{1}{1+\mu_r/2} z,\\ \Phi_M^{\rm out} &= Ia^2\frac{\mu_r/2}{1+\mu_r/2}\frac{1}{r^2}\cos\theta = Ia^2\frac{\mu_r/2}{1+\mu_r/2}\frac{z}{r^3}\end{aligned}$$

This gives rise to a magnetic induction

$$\begin{aligned}\vec{B}^{\rm in} &= \mu\frac{I}{a}\frac{1}{1+\mu_r/2}\hat{z}\\ \vec{B}^{\rm out} &= \mu_0 I a^2 \frac{\mu_r/2}{1+\mu_r/2}\frac{3(z/r)\hat{r}-\hat{z}}{r^3}\end{aligned}$$

The interior field is uniform, while the exterior field is that of a magnetic dipole.

[25 pts] 3. A semi-infinite coaxial cable consists of an inner conductor of radius a surrounded by an outer conductor of radius b. A dielectric with permittivity ϵ and permeability μ fills the volume between the conductors.



[5] a) If a constant (static) potential difference V_0 is applied between the conductors, what is the electric field inside the cable? Ignore fringe effects.

It is natural to use cylindrical coordinates for this problem. For the electrostatics problem, an elementary application of Gauss' law gives an electric field

$$\vec{E} = \frac{1}{2\pi\epsilon}\frac{\lambda}{\rho}\hat{\rho}$$

where λ is the charge per unit length on the inner conductor. Since the potential difference between conductors is V_0, we have

$$V_0 = -\int_a^b \vec{E}\cdot d\vec{\ell} = -\int_a^b \frac{\lambda}{2\pi\epsilon}\frac{d\rho}{\rho} = -\frac{\lambda}{2\pi\epsilon}\log\left(\frac{b}{a}\right)$$

As a result, the electric field is given in terms of V_0 by

$$\vec{E} = -\frac{V_0}{\rho\log(b/a)}\hat{\rho}$$

[15] b) Show that, if a sinusoidal potential difference $V(t) = V_0 e^{-i\omega t}$ is applied at the end of the cable, then Maxwell's equations admit a traveling wave solution

$$\vec{B} = \hat{\phi}\mathcal{B}(\rho)e^{i(kz-\omega t)}, \qquad \vec{E} = \hat{\rho}\mathcal{E}(\rho)e^{i(kz-\omega t)}$$

where z is the direction along the axis of the cable. Find $\mathcal{B}(\rho)$ and $\mathcal{E}(\rho)$ in terms of V_0.

The sinusoidal potential difference is of harmonic form. Thus we may examine the harmonic Maxwell's equations. Firstly, Gauss' law for magnetism, $\vec{\nabla}\cdot\vec{B} = 0$, is trivially satisfied for the above solution. For the source-free Gauss' law, $\vec{\nabla}\cdot\vec{E} = 0$, we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho\mathcal{E}(\rho)\big) = 0 \qquad \Rightarrow \qquad \mathcal{E} = \frac{C}{\rho}$$

for some constant C. This allows us to write

$$\vec{E} = \hat{\rho}\frac{C}{\rho}e^{i(kz-\omega t)}$$

Faraday's law, $\vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$, then gives

$$ik\frac{C}{\rho} - i\omega\mathcal{B} = 0 \qquad \Rightarrow \qquad \mathcal{B} = \frac{C}{\rho}\frac{k}{\omega}$$

so that

$$\vec{B} = \hat{\phi}\frac{C}{\rho}\frac{k}{\omega}e^{i(kz-\omega t)}$$

The remaining equation to verify is the Ampère-Maxwell equation, $\vec{\nabla}\times\vec{B} + i\mu\epsilon\omega\vec{E} = 0$, which gives

$$-i\frac{C}{\rho}\frac{k^2}{\omega} + i\frac{C}{\rho}\mu\epsilon\omega = 0 \qquad \Rightarrow \qquad k = \sqrt{\mu\epsilon}\,\omega$$

As a result, Maxwell's equations are solved provided we impose the standard dispersion relation $k = \sqrt{\mu\epsilon}\,\omega$. Note that, if we take $z = 0$ to be the end of the cable, we may solve for the constant C by imposing

$$V_0 e^{-i\omega t} = -\int_a^b \vec{E}|_{z=0}\cdot d\vec{l} = -\int_a^b \frac{C}{\rho}e^{-i\omega t}d\rho = -Ce^{-i\omega t}\log\left(\frac{b}{a}\right)$$

This gives $C = -V_0/\log(b/a)$, so that

$$\mathcal{E}(\rho) = -\frac{V_0}{\rho\log(b/a)}, \qquad \mathcal{B}(\rho) = -\frac{\sqrt{\mu\epsilon}\,V_0}{\rho\log(b/a)}$$

Note that traveling waves in the $-z$ direction (as well as superpositions of waves) are also possible.

[5] *c)* What is the impedance Z (given by the complex Ohm's law, $V = IZ$) of the cable?

The impedance is given by $Z = V/I$. The potential at the end of the cable ($z = 0$) is already given, so all we need is the current. The current may be obtained from Ampère's law in integral form

$$\oint \vec{B}\cdot d\vec{l} = \mu I$$

where we integrate along a circle of radius $a < \rho < b$ located at $z = 0$. This gives

$$I = \frac{1}{\mu}\frac{\sqrt{\mu\epsilon}\,V_0}{\rho\log(b/a)}\times(2\pi\rho) = 2\pi\sqrt{\frac{\epsilon}{\mu}}\frac{V_0}{\log(b/a)}$$

The impedance is then

$$Z = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\log\left(\frac{b}{a}\right)$$

which is real and independent of frequency. This is a feature of coaxial transmission lines. Note that the displacement current term is in the $\hat{\rho}$ direction (since this is the direction of the electric field) and does not contribute to the above application of Ampère's law. Alternatively, we may calculate the current using the matching condition $\hat{n} \times \vec{H}|_S = \vec{K}$ (where S denotes the surface of the conductor) to obtain the surface current density $\vec{K}$ and they by $I =$ (circumference) $\times K$.

[25 pts] 4. A plane polarized electromagnetic wave of frequency ω in free space is normally incident on the flat surface of an excellent conductor ($\mu = \mu_0$, $\epsilon = \epsilon_0$ and $\sigma \gg \omega\epsilon_0$) which fills the region $z > 0$. Assume the incident wave is given by $\vec{E} = \hat{x}E_i e^{i(kz-\omega t)}$

[10] *a*) What is the current density $\vec{J}$ inside the conductor (in the limit $\sigma \gg \omega\epsilon_0$)? Express your result in terms of the skin depth $\delta = \sqrt{2/\mu_0\sigma\omega}$.

For a normally incident plane wave, we take the incoming wave to be in a medium with index of refraction n and the transmitted wave to be in a medium with index of refraction n'. Then

$$\text{incident:} \quad \vec{E} = \hat{x}E_i e^{i(kz-\omega t)}$$
$$\text{transmitted:} \quad \vec{E}' = \hat{x}E_i \left(\frac{2n}{n+n'}\right) e^{i(k'z-\omega t)}$$

For this problem, we have $n = 1$. For n', we use the excellent conductor approximation

$$n' = \sqrt{1 + i\frac{\sigma}{\omega\epsilon_0}} \approx \sqrt{i\frac{\sigma}{\omega\epsilon_0}} = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\epsilon_0}} = (1+i)\frac{c}{\delta\omega}$$

where $\delta = \sqrt{2/\mu_0\sigma\omega}$ is the skin depth. The transmitted wavenumber k' is

$$k' = \frac{\omega}{c}n' \approx \frac{1+i}{\delta}$$

Hence

$$\vec{E}' = \hat{x}E_i \frac{2}{1+n'}e^{i(k'z-\omega t)} \approx \hat{x}E_i\frac{2}{n'}e^{i(k'z-\omega t)} \approx \hat{x}E_i(1-i)\frac{\delta\omega}{c}e^{i(z/\delta-\omega t)}e^{-z/\delta}$$

The current density inside the conductor is then

$$\vec{J} = \sigma\vec{E}' \approx \hat{x}E_i(1-i)\frac{\delta\omega\sigma}{c}e^{i(z/\delta-\omega t)}e^{-z/\delta} = \hat{x}\sqrt{\frac{\epsilon_0}{\mu_0}}E_i(1-i)\frac{2}{\delta}e^{i(z/\delta-\omega t)}e^{-z/\delta} \quad (4)$$

[5] *b*) Now assume that the conductor is perfect. Solve for the reflected wave in the limit of a perfect conductor. (Note that $E^{\|}$ vanishes at the surface of a perfect conductor.)

If a wave $\vec{E} = \hat{x}E_i e^{i(kz-\omega t)}$ is normally incident on a perfect conductor, the reflected wave will have the form $\vec{E}'' = \hat{x}E'' e^{i(-kz-\omega t)}$. The total electric field at $z = 0$ (the surface of the conductor) is then $\hat{x}(E_i + E'')e^{-i\omega t}$. Since this is in the parallel direction, it must vanish. Hence $E'' = -E_i$. The reflected wave is then

$$\vec{E}'' = -\hat{x}E_i e^{i(-kz-\omega t)}$$

This is interpreted as a 180° phase shift.

[10] *c*) Compute the idealized surface current density $\vec{K}$ on the surface of the perfect conductor, and show that it satisfies the relation

$$\vec{K} = \int_0^\infty \vec{J}\,dz$$

where $\vec{J}$ is the current density found in part a.

The surface current density is given by

$$\vec{K} = \hat{z} \times (\vec{H}' - \vec{H} - \vec{H}'')\big|_{z=0} = -\frac{1}{\mu_0}\hat{z} \times (\vec{B} + \vec{B}'')\big|_{z=0}$$

where we used the fact that the transmitted wave $\vec{B}'$ vanishes in a perfect conductor. The incident and reflected magnetic inductions are

$$\vec{B} = \sqrt{\mu_0\epsilon_0}\hat{z} \times \vec{E}, \qquad \vec{B}'' = \sqrt{\mu_0\epsilon_0}(-\hat{z}) \times \vec{E}''$$

Hence

$$\vec{K} = -\sqrt{\frac{\epsilon_0}{\mu_0}}\hat{z} \times \left(\hat{z} \times (\vec{E} - \vec{E}'')\right)\big|_{z=0} = \sqrt{\frac{\epsilon_0}{\mu_0}}(\vec{E} - \vec{E}'')\big|_{z=0}$$

Substituting in $\vec{E}$ and $\vec{E}''$ gives

$$\vec{K} = \hat{x}\sqrt{\frac{\epsilon_0}{\mu_0}}(2E_i)e^{-\omega t}$$

We now compare this with the current density found in part a. From (4) we have

$$\begin{aligned}\int_0^\infty \vec{J}dz &= \hat{x}\sqrt{\frac{\epsilon_0}{\mu_0}}E_i(1-i)\frac{2}{\delta}e^{-i\omega t}\int_0^\infty e^{-(1-i)z/\delta}dz \\ &= \hat{x}\sqrt{\frac{\epsilon_0}{\mu_0}}E_i(1-i)\frac{2}{\delta}e^{-i\omega t}\frac{\delta}{1-i} \\ &= \hat{x}\sqrt{\frac{\epsilon_0}{\mu_0}}2E_i e^{-i\omega t}\end{aligned}$$

So we see that $\vec{K} = \int_0^\infty \vec{J}\,dz$ is indeed satisfied.