Physics 505 Fall 2003

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December 18, 2003

Disclaimer: The purpose of these notes is to provide you with a general list of topics that were covered in class. The notes are not a substitute for reading the textbook, nor is it guaranteed that they are complete. If you find typos, please report them to me.

1 9/2/2003

Units in Jackson. Gaussian and SI units were discussed. The equations for the electric field of a point charge and the magnetic field of a line current were written down for both systems. Units of charge, length, E-field, B-field, current, time, and force were discussed. Pages 781ff of the textbook were pointed out.

Example. Fine-structure constant in Gauss units, $\alpha = \frac{e^2}{\hbar c}$, equals fine-structure constant in SI units, $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$. The explicit calculations yielding $\frac{1}{137}$ in both systems (unit-less) were sketched.

The fine-structure formula for the energy levels of hydrogen-like ions was presented. The role of α^2 as a measure for the strength of EM interactions was discussed. The diagram of the lowest H-levels ($n = 1$ and 2) was shown and various effects (FS, Lamb-shift) were addressed.

Range of validity of Coulomb's law. Qualitative limits are given by Lamb-shift QED calculations, which indicate deviations from Coulomb's law at lengths $< \frac{\lambda_e}{2\pi} = \frac{\hbar}{m_e c} = \alpha\, a_0 = 0.4\ pm$. A lower limit for the validity range of Coulomb's law is given by the order of the earth's radius.

Math background. Some pieces of vector calculus were reviewed, including three integral theorems, parametrization of line and area integrals, coordinate transformation in volume integrals, Jacobi determinant.

(see http://www.ima.umn.edu/ esedoglu/teaching/math-2374/example1/example1.html for some examples).

Basic laws. Coulomb's law for force and electric field. Superposition principle. E-field of a continuous charge distribution $\rho(\mathbf{x})$. Atomic electric-field unit.

Reading assignment: Section about the δ-function.

2 9/4/2003

Gauss's law. Geometrical derivation using the $1/r^2$-dependence of the field of a point charge and the

superposition principle. Analytical derivation of the differential form of the law by evaluating

$$\nabla_x \cdot \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' = ... = \frac{\rho(\mathbf{x})}{\epsilon_0} \,. \tag{1}$$

The derivation has also provided the following insights:

$$\begin{aligned}
\nabla_x \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= -\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\
\nabla_{x'} \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\
\nabla_x \cdot \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} &= 4\pi\delta(\mathbf{x}-\mathbf{x}') \\
\nabla_{x'} \cdot \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} &= -4\pi\delta(\mathbf{x}-\mathbf{x}') \\
\nabla_x^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \nabla_{x'}^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')
\end{aligned} \tag{2}$$

Electrostatic potential. Brief review.

Boundary conditions for field **E** and potential Φ on charged sheets and layers of dipoles.

Poisson and Laplace equation.

Derivation of Green's identities.

Boundary conditions for determination of $\Phi(\mathbf{x})$. Dirichlet and Neumann boundary conditions. Derivation of Uniqueness theorems.

3 9/9/2003

In context with Jackson, Problem 1.11, it was shown that any curved surface can be locally parametrized using a function

$$H(x', y') = \tilde{\alpha} x'^2 + \tilde{\beta} y'^2 = \frac{1}{2R_1} x'^2 + \frac{1}{2R_2} y'^2 \,. \tag{3}$$

Review of uniqueness theorems for Dirichlet and Neumann boundary conditions, and for a situation of n conductors with specified charges Q_n on them.

Integral relation from Green I:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' + \frac{1}{4\pi} \int_{\partial V} \{ \frac{1}{|\mathbf{x}-\mathbf{x}'|} \frac{\partial}{\partial n'} \Phi(\mathbf{x}') - \Phi(\mathbf{x}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \} da' \tag{4}$$

The uselessness of this relation was discussed.

Green's function for Dirichlet Boundary Conditions (BC):

$$\begin{aligned} \Delta' G(\mathbf{x},\mathbf{x}') &= -4\pi\delta(\mathbf{x}-\mathbf{x}') \quad \forall \quad \mathbf{x}' \in V \\ G(\mathbf{x},\mathbf{x}') &= 0 \quad \forall \quad \mathbf{x}' \in \partial V \end{aligned} \tag{5}$$

is, by the uniqueness theorem, uniquely solved via

$$\begin{aligned} F(\mathbf{x},\mathbf{x}') &:= G(\mathbf{x},\mathbf{x}') - \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ \Delta' F(\mathbf{x},\mathbf{x}') &= 0 \quad \forall \quad \mathbf{x}' \in V \\ F(\mathbf{x},\mathbf{x}') &= -\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \forall \quad \mathbf{x}' \in \partial V \end{aligned} \tag{6}$$

The significance of $\mathbf{x}$ and $\mathbf{x}'$ (parameter / variable of PDE) was discussed. The universality of G, which only depends on the geometry, was pointed out.

The generality and usefulness of the resultant solution for $\Phi(\mathbf{x})$ was discussed:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\int_{\partial V}\Phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \tag{7}$$

A interpretation of $G(\mathbf{x},\mathbf{x}')$ and of the auxiliary function $F(\mathbf{x},\mathbf{x}')$ in terms of image charges was provided.

Green's function for Neumann BC:

The corresponding formalism for Neumann BC is:

$$\begin{aligned} \Delta' G(\mathbf{x},\mathbf{x}') &= -4\pi\delta(\mathbf{x}-\mathbf{x}') \quad \forall \quad \mathbf{x}' \in V \\ \frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}') &= -\frac{4\pi}{S} \quad \forall \quad \mathbf{x}' \in \partial V\,, \end{aligned} \tag{8}$$

where S is the area of ∂V. This boundary value problem for G is, by the uniqueness theorems, solved via

$$\begin{aligned} F(\mathbf{x},\mathbf{x}') &:= G(\mathbf{x},\mathbf{x}') - \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ \Delta' F(\mathbf{x},\mathbf{x}') &= 0 \quad \forall \quad \mathbf{x}' \in V \\ \frac{\partial}{\partial n'}F(\mathbf{x},\mathbf{x}') &= -\frac{4\pi}{S} - \frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \forall \quad \mathbf{x}' \in \partial V \end{aligned} \tag{9}$$

yielding

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\int_{\partial V} G(\mathbf{x},\mathbf{x}')\frac{\partial}{\partial n'}\Phi(\mathbf{x}')da' + \langle\Phi(\mathbf{x}')\rangle_{\partial V} \tag{10}$$

with the average value of Φ on ∂V denoted $\langle\Phi(\mathbf{x}')\rangle_{\partial V}$. The usefulness of this solution for $\Phi(\mathbf{x})$ for "exterior problems" with $S \to \infty$ and $\Phi(x \to \infty) = 0$ was discussed.

Electrostatic energy: Basic equations for the energy of a set of point charges, of a continuous charge distribution $\rho(\mathbf{x})$ with potential $\Phi(\mathbf{x})$, and of a given electric field $\mathbf{E}(\mathbf{x})$ were reviewed. The self-energy contribution in the latter two was pointed out. It was noted that the energy density related to an electric field leads to an electrostatic pressure, $\mathbf{p} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$, on charged conductor surfaces.

Using the uniqueness theorem for charged conductors, it was shown that a set of n conductors with potentials V_i and charges Q_i satisfies the **capacitor equation** $V_j = \sum_{i=1}^{n} p_{j,i}Q_i$ and its inverse, $Q_i = \sum_{j=1}^{n} C_{i,j}V_j$. The reasoning for the nomenclature for the capacitances $C_{i,i}$ and the induction coefficients $C_{i,j}$ with $i \neq j$ was explained. Expressions for the energy were given:

$$W = \frac{1}{2}\sum_{i,j=1}^{n} p_{i,j}Q_iQ_j = \frac{1}{2}\sum_{i,j=1}^{n} C_{i,j}V_iV_j \tag{11}$$

A brief review of how to calculate the capacitance of a conductor pair was provided (assume charges $\pm Q$, find $\mathbf{E}(\mathbf{x})$, calculate $V = -\int_1^2 \mathbf{E}(\mathbf{x}) \cdot \mathbf{dl}$, then $C = Q/V$).

4 9/11/2003

Some questions concerning the proof of the capacitor equation, $V_i = \sum_{j=1}^{n} p_{i,j}Q_j$, which is the only non-trivial part of the discussion on page 43 of the textbook, have been clarified.

Variational principles. It has been shown that the functional

$$I[\psi] = \frac{1}{2}\int_V |\nabla\psi|^2 d^3x - \int_V \frac{\rho(\mathbf{x})}{\epsilon_0}\psi d^3x \tag{12}$$

becomes minimal if the test function $\psi(\mathbf{x})$ satisfies the Poission Equation and Dirichlet BC.

The benefit is two-fold: firstly, it is seen that a variational principle exists that leads to the Poission Equation. Secondly, it follows that variational methods can be used to find approximate solutions to the electrostatic boundary-value problem. To see this, consider a given set of test functions $\{\psi_{\alpha,\beta,...}(\mathbf{x})\}$ satisfying the Dirichlet BC of the problem, with parameters $\alpha, \beta, ...$ Then, within the chosen set of test functions the function $\psi_{\alpha_0,\beta_0,...}(\mathbf{x})$ for which $I[\psi]$ becomes minimal represents the best approximation to the actual solution $\Psi(\mathbf{x})$. To identify $\psi_{\alpha_0,\beta_0,...}(\mathbf{x})$, calculate $I(\alpha,\beta,...) := I[\psi_{\alpha,\beta,...}]$ and find the parameters $(\alpha_0, \beta_0, ...)$ for which $\frac{\partial}{\partial\alpha}I(\alpha,\beta,...) = 0$, $\frac{\partial}{\partial\beta}I(\alpha,\beta,...) = 0$ etc., and for which the value of I is lowest.

A functional suitable for Neumann BC has been given. The class has been advised to read the examples provided in the textbook.

Relaxation methods. For charge-free volumes with Dirichlet BC, the "cross-averaging method" and its foundation in the previously discussed variational principle has been explained. The Jacobi and the Gauss-Seidel iteration methods have been shown. A hyper-relaxation method yielding significantly improved convergence has been presented.

The accuracy of cross-average relaxation method in terms of the grid size has been derived.

Improved methods that use weighted cross- and square-averages and a method that allows for the incorporation of non-zero charge densities have been pointed out.

5 9/16/2003

The **connection between the image charge method and the Dirichlet Green's function** G_D has been explained. Assume that a charge q at location $\mathbf{x}'$ in the volume of interest produces image charges $q_i(\mathbf{x}')$ at locations $\mathbf{x}_i$ outside the volume of interest. Since the image charges are proportional to q, we may write $q_i(\mathbf{x}') = q\,\tilde{q}_i(\mathbf{x}')$, with the normalized image charges $\tilde{q}_i(\mathbf{x}')$ measuring the values of the image charges relative to q. Then,

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \sum_i \frac{\tilde{q}_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}_i(\mathbf{x}')|} \tag{13}$$

The equations and boundary conditions for G_D and F_D are:

$$\begin{aligned} \Delta' G(\mathbf{x},\mathbf{x}') &= -4\pi\delta(\mathbf{x}-\mathbf{x}') \quad \forall \quad \mathbf{x}' \in V \\ G(\mathbf{x},\mathbf{x}') &= 0 \quad \forall \quad \mathbf{x}' \in \partial V \end{aligned} \tag{14}$$

and

$$\begin{aligned} \Delta' F(\mathbf{x},\mathbf{x}') &= 0 \quad \forall \quad \mathbf{x}' \in V \\ F(\mathbf{x},\mathbf{x}') &= -\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \forall \quad \mathbf{x}' \in \partial V \quad . \end{aligned} \tag{15}$$

These equations make it evident that F_D plays the role of the potential of image charges generated by a "charge" $4\pi\epsilon_0$.

An important consequence of the connection between Green's function and image charges is that once a Dirichlet problem has been solved with the image charge method, its Green's function is also known, and a larger class of problems can be solved.

Example. The case of a conducting plane at $x = 0$ and the volume of interest being the half-space $x > 0$ has been discussed. A charge q at location $\mathbf{x}'$ produces one image charge $q_1 = -q$ at location $\mathbf{x}_1(\mathbf{x}') = \mathbf{x}' - (2\mathbf{x}'\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}$. The normalized value of the image charge is -1, and consequently

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'+(2\mathbf{x}'\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}|} \quad . \tag{16}$$

According to the Green's function formalism for Dirichlet BC, it is

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_{\text{volume } x'>0} \rho(\mathbf{x}') \left\{ \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'+(2\mathbf{x}'\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}|} \right\} dx'dy'dz' \\ &+ \frac{1}{4\pi}\int_{\text{plane } x'=0} \Phi(0,y',z') \frac{\partial}{\partial x'} \left\{ \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'+(2\mathbf{x}'\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}|} \right\} dy'dz' \end{aligned} \tag{17}$$

Note that in this problem $\frac{\partial}{\partial n'} = -\frac{\partial}{\partial x'}$, and that x' is the cartesian x-coordinate of $\mathbf{x}'$ (not the radial coordinate of $\mathbf{x}'$, as in numerous other equations in the textbook).

The first line in Eq. 17 represents the term that immediately follows from the image charge method and the superposition principle. The second line requires knowledge of the Green's function formalism, and

allows one to treat the extended class of problems in which an arbitrary potential on the surface ∂V is specified.

The image-charge problem of a **charge q outside a grounded, conducting sphere** with radius a has been discussed. We have obtained the potential $\Phi(\mathbf{x})$, the induced surface charge density σ on the sphere, the total induced charge, and the forces on the charge and the sphere. The superposition principle allows for some straightforward extensions. These include the cases of a charge q outside a conducting sphere with a given potential V or with a given total charge Q.

The basic method is explained in the following for fixed V. We identify quantities obtained for the case of a charge q outside the grounded sphere with a subscript I. Consider the solutions of the following problems: I=grounded sphere with charge q outside. II=sphere on potential V and no charge outside the sphere. The solution of case II is a constant potential V on and inside the sphere, and $\Phi_{II}(\mathbf{x}) = \frac{Va}{|\mathbf{x}|}$ outside the sphere, with a total charge of $4\pi\epsilon_0 Va$ evenly distributed over the surface of the sphere. The sum of the charge densities of cases I and II, and the sum of the potentials, $\Phi = \Phi_I + \Phi_{II}$, satisfy the boundary conditions. That is: the sums produce the correct external charge distribution and the correct potential on the boundary, respectively. Also, due to the superposition principle, the sum potential and the sum charge distribution are a solution of the Poisson equation. Due to the uniqueness theorem, this must be the only solution for the given surface potential V and the given charge distribution in the volume of interest.

Thus, outside the sphere it is $\Phi = \Phi_I + \Phi_{II} = \Phi_I + \frac{Va}{|\mathbf{x}|}$. The surface charge density on the surface is $\sigma = \sigma_I + \sigma_{II} = \sigma_I + \frac{\epsilon_0 V}{a}$, and the total charge is $-q + 4\pi\epsilon_0 Va$. Since the electric fields also follow the superposition principle, the force is $\mathbf{F} = \mathbf{F}_I + \frac{qVa}{y^2}\hat{\mathbf{y}}$.

Sometimes, simple tricks allow for the treatment of seemingly unrelated problems. In the present instance, the problem of a charge q outside a grounded, conducting sphere can be easily twisted in a way that allows for a simple calculation of the potential and the charge densities of a conducting sphere in a homogeneous electric field. **Read textbook**.

The Dirichlet Green's function of a spherical surface has been deduced form the corresponding image-charge problem. In a system of spherical coordinates with z-axis pointing to the location of interest $\mathbf{x}$, and denoting the angle between the vectors $\mathbf{x}$ and $\mathbf{x}'$ by γ, the Green's function is seen to be

$$G_D(\mathbf{x},\mathbf{x}') = G_D(x,x',\cos\gamma) = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma}} \quad , \tag{18}$$

which is symmetric (as must be), and on the surface it is

$$\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')|_{x'=a} = -\frac{\partial}{\partial x'}G_D(x,x',\cos\gamma)|_{x'=a} = -\frac{x^2-a^2}{a\sqrt{x^2+a^2-2ax\cos\gamma}^3} \quad . \tag{19}$$

Noting that the spherical coordinates of the source location in the chosen frame are (x',γ,ϕ'), for the calculation of the potential one writes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_{\text{volume } x'>a} \rho(x',\gamma,\phi')G_D(x,x',\cos\gamma)x'^2 dx' d\phi' d\cos\gamma + \frac{1}{4\pi}\int_{\text{surface } x'=a} \Phi(a,\gamma,\phi')\frac{a(x^2-a^2)}{\sqrt{x^2+a^2-2ax\cos\gamma}^3} d\phi' d\cos\gamma \quad . \tag{20}$$

It was shown that $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. To disentangle the reference frame from the point of interest $\mathbf{x}$, one can then write

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_{\text{volume } x'>a} \rho(x',\theta',\phi')G_D(x,x',\cos\gamma(\theta,\theta',\phi,\phi'))x'^2 dx' d\cos\theta' d\phi' + \frac{1}{4\pi}\int_{\text{surface } x'=a} \Phi(a,\theta',\phi')\frac{a(x^2-a^2)}{\sqrt{x^2+a^2-2ax\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\right]}^3} d\phi' d\cos\theta' \tag{21}$$

The class has been advised to read the example of **two hemispheres on opposite potentials** discussed in the textbook.

6 9/18/2003

Basic concepts of electrostatics have been reviewed: Uniqueness theorems, Green's function, relations between Green's function and image charges.

The problem of **two hemispheres with radius a on potentials V and $-V$ in a charge-free space** has been briefly discussed. The resultant potential follows from the surface part of Eq. 21, and is

$$\Phi(\mathbf{x}) = \frac{Va(x^2-a^2)}{4\pi}\int_{\phi'=0}^{2\pi} d\phi' \int_{\cos\theta'=-1}^{1} d\cos\theta' \left\{ \frac{1}{\sqrt{x^2+a^2-2ax\cos\gamma}^3} - \frac{1}{\sqrt{x^2+a^2+2ax\cos\gamma}^3} \right\}, \tag{22}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. The integral can be calculated along the $+z$-axis, where $\gamma = \theta'$.

Two methods of how to obtain an approximate solution for $x \gg a$, valid for all θ, have been pointed out:

1) The exact potential can be expanded along the $+z$-axis, yielding an expression of the form $\Phi(z) = \sum_{n=1,3,\ldots} a_n z^{-n-1}$ with expansion coefficients a_n. In the given problem, the coefficients a_n for even n vanish because of symmetry. The general potential is then given by $\Phi(x,\theta) = \sum_{n=1,3,\ldots} a_n x^{-n-1} P_n(\cos\theta)$. This method will be further discussed later.

2) The square root under the integral may be expanded in terms of a small parameter ϵ that involves $\cos\theta$. One may write, for instance,

$$(x^2+a^2-2ax\cos\gamma)^{-3/2} = (x^2+a^2)^{-3/2}(1-\epsilon)^{-3/2} \quad \text{with} \quad \epsilon = \frac{2ax\cos\gamma}{x^2+a^2} \ll 1 \tag{23}$$

The expansion in ϵ leads to solvable integrals, and, eventually, to a power series involving odd powers of $\cos\theta$.

The **general strategy of solving the Laplace equation using the variable separation method** has been outlined. Usually, there are three main steps involved:

1) Find a complete set of orthogonal functions that solve the Laplace equation and fit the given geometry and the BC. This can often be accomplished using the method of variable separation, which usually involves the following steps:

1a) Consider the symmetry of problem to make the best choice of a coordinate system. Box problems and other problems involving (mostly) right angles and straight surfaces are treated with cartesian coordinates. Problems with circles, circle segments, cylinders etc. are usually treated in cylindrical coordinates, and problems involving spheres or sections of spheres with spherical coordinates.

1b) Write $\Delta\Phi = 0$ in the coordinates identified in step 1a).

1c) Write down a set of solutions obtained from the variable separation method. Completeness may matter.

1d) Elimination process. To simplify the further steps, use "simple" boundary conditions to reduce the number of basis functions and expansion coefficients from the result of 1c). "Simple" BC are, for instance, surface portions that are entirely on zero potential. Often, the formation of certain linear combinations helps (see, for instance, the $\sinh(\gamma z)$ term in Eq. 2.53). Eliminate diverging terms as appropriate.

2) Write the potential as a linear superposition of the functions left over after step 1d).

3) Find the expansion coefficients by surface integrals over surfaces with known boundary values. Use the orthogonality of the complete set of functions.

As a specific example, **a rectangular three-dimensional box with five faces on zero potential and the upper xy-face on a potential $V(x, y)$** was discussed. This problem is treated using cartesian coordinates. The proper use of sin, cos, exp, sinh and cosh for the basis functions was explained. It was explained why the superposition principle allows one to generalize the result to arbitrary Dirichlet BC on the box (all sides on arbitrary potential, not just one).

As another example, the **two-dimensional Laplace equation has been separated in cylindrical coordinates**, and proper basis solutions for the case of a two-dimensional grounded corner or edge have been obtained.

In addition to your regular reading of the textbook, make sure you take special care of the following **Reading assignments**:

- Refresh undergraduate knowledge of the separation method. If necessary, consult an undergraduate textbook such as Griffiths, Introduction to Electrodynamics.
- Read section 2.8 in the textbook (about orthogonal function sets). Review the orthonormality and the completeness properties, and the given examples (discrete Fourier series, Fourier transform...).
- Read the portions of sections 2.9 - 2.11 that were skipped in class.

7 9/25/2003

The separation method has been reviewed. In connection with Chapter 2.10 of the textbook, the relation between the **Laplace Equation and analytic functions** has been discussed. Proofs for the following facts have been outlined:

- For any analytic function $f(z) = u(x, y) + \mathrm{i}v(x, y)$, with $z = x + \mathrm{i}y$, the Cauchy-Rieman equations apply: $\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}$ and $\frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y}$.
- $\Delta u = \Delta v = 0$.

Therefore, the imaginary and reals parts of analytic functions often coincide with the solutions of 2D potential problems.

More examples were discussed:

- The separation method has been used to treat the 2D potential problem of corners and edges with straight surfaces on a constant potential V_0 and a potential $V(\phi)$ on the cylindrical section of the surface. The solution is

$$\tilde{\Phi}(\rho, \phi) = \sum_{n=1}^{\infty} a_n \rho^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi}{\beta}\phi\right) \quad \text{with} \quad a_n = \frac{2}{\beta R^{\frac{n\pi}{\beta}}} \int_{\phi=0}^{\beta} \tilde{V}(\phi) \sin\left(\frac{n\pi}{\beta}\phi\right) d\phi \tag{24}$$

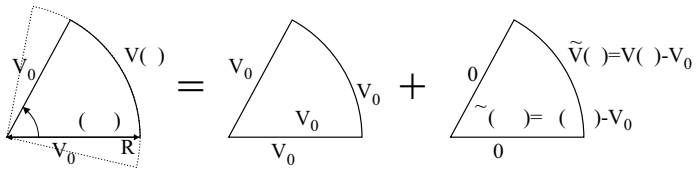

Figure 1: 2D problem of a corner on potential V_0 on the straight sections of the surface and potential $V(\phi)$ on the cylindrical section. The figure shows how to use the superposition principle to reduce the problem to the case of zero potential on the straight surfaces.

  As example illustrating the use of the results, we have estimated the outcome of a molecular-beam deflection experiment in which molecules with a linear polarizability are deflected when traveling parallel to the edge of a thin, grounded conducting plate surrounded by a cylinder on a potential V.

- Two concentric cylinders with radii $a < b$ and surface potentials $V_a(\phi)$ and $V_b(\phi)$ on the surfaces. Discussion of the general solution

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \rho + \sum_{n=\pm 1, \pm 2, \ldots}^{\pm\infty} \left(a_n \rho^n + b_n \rho^{-n}\right) \left(A_n \cos(n\phi) + B_n \sin(n\phi)\right) \quad , \tag{25}$$

  which is analogous to Eq. 2.71 in the textbook. (issues: which terms to drop in what case, how to find the remaining coefficients).

Finite element method. The 2D case of finite-element functions of pyramidal shape that are arranged on a square grid has been discussed.

Reading: improved method using triangular shape functions.

8 9/30/2003

Solution of $\Delta\Phi = 0$ by **separation of variables in spherical coordinates**: It has been outlined how the ϕ-, the r- and the θ-equations are obtained and solved. The solution of the θ-equation has been limited, for now, to the case of azimuthal symmetry ($m = 0$ all over).

One may add that for the ϕ- and r-equations and for given values of the separation variables m^2 and $l(l+1)$ *two* linearly independent solutions contribute, which is the maximum number possible because the underlying linear homogeneous differential equations are of 2nd order. For the θ-equation *only one* solution $P_l(\cos\theta)$ is considered as *physically relevant* solution. The other solution, which formally exists, is obtained by making the other possible choice of α (textbook, Eqn. 3.13, 1st line). That other solution diverges at $x = \cos\theta = \pm 1$ and is therefore discarded.

Reading: Properties of Legendre polynomials.

Orthogonality: $\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{l\,l'}$

Function Expansions: From orthogonality and completeness it follows that for well-behaved $f(x)$ on $[-1,1]$ it is $f(x) = \sum_{l=0,1,..}^{\infty} A_l P_l(x)$ with $A_l = \frac{2l+1}{2}\int_{-1}^{1} P_l(x)f(x)\,dx$.

Closure: It has been explained how to obtain the closure relation $\delta(x-x') = \sum_{l=0,1,..}^{\infty} \frac{2l+1}{2} P_l(x)P_l(x')$.

The use of **recursion relations between P_l's** has been briefly discussed.

Potential boundary value problems with azimuthal symmetry: The potential is of the general form $\Phi(r,\theta) = \sum_{l=0,1,..}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$, where the A_l and B_l are determined from BC.

Example: For given potential $V(\theta)$ on a sphere with radius a, the interior solution has $B_l = 0$ and $A_l = \frac{2l+1}{2a^l}\int_{-1}^{1} V(\theta)P_l(\cos\theta)\,d\cos\theta$. (I believe I had the factor $\frac{2l+1}{2}$ up-side-down when I wrote this on the board - if so please correct in your notes). For the exterior solution, $A_l = 0$ and $B_l = \frac{(2l+1)a^{l+1}}{2}\int_{-1}^{1} V(\theta)P_l(\cos\theta)\,d\cos\theta$.

Another **Method of obtaining the expansion coefficients** A_l and B_l is to perform an exact calculation of $\Phi(z)$ along the z-axis, where $P_l \equiv 1$. Then, expand $\Phi(z)$ in either negative or positive powers of z, dependent on whether you seek an exterior or interior solution, respectively. For an interior solution, you may find, for instance, $\Phi(z) = \sum_{l=0,1,..}^{\infty} A_l z^l$. If the problem has azimuthal symmetry, then the potential at a general location is $\Phi(r,\theta) = \sum_{l=0,1,..}^{\infty} A_l r^l P_l(\cos\theta)$, with the same coefficients A_l as in the expansion of Φ along the z-axis. The reason why the method works is the uniqueness of the expansion of both $\Phi(z)$ along the z-axis and $\Phi(r,\theta)$.

As an application, we have derived the following **important expansion** of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0,1,...}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) \quad , \quad \text{where} \quad r_< = \min(r,r') \quad \text{and} \quad r_> = \max(r,r') \tag{26}$$

As an application of the expansion Eq. 26, we have expanded the potential of a ring charge Q that is concentric with the z-axis, has a radius a, and has a height b above the xy-plane. This example is covered in the textbook on page 103. What's most important to note about this example is that the use of the expansion Eq. 26 saves us from having to expand $\frac{1}{\sqrt{z^2+c^2-2cz\,\cos\alpha}}$ as an infinite power series of z:

The **hard way** to find $\Phi(r,\theta)$ would be to first write $\Phi(z,\theta=0) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+c^2-2cz\cos\alpha}}$, to then

expand this either in powers of z for small z or in powers of $\frac{1}{z}$ for large z, to write the result in the form $\Phi(z) = \sum_{l=0,1,..}^{\infty} A_l z^l$ or $\Phi(z) = \sum_{l=0,1,..}^{\infty} B_l z^{-l-1}$, and to write the general potential as $\Phi(r,\theta) = \sum_{l=0,1,..}^{\infty} A_l r^l P_l(\cos\theta)$ or $\Phi(r,\theta) = \sum_{l=0,1,..}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$. It will finally be left to show that the coefficients can be written in the concise form $A_l \frac{1}{c^{l+1}} P_l(\cos\alpha)$ and $B_l = c^l P_l(\cos\alpha)$, respectively.

Easy way: The use of Eq. 26 yields, after a few lines of just writing down facts rather than doing calculations, $\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0} \sum_{l=0,1,...}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$, where $r_< = \min(r,c)$ and $r_> = \max(r,c)$.

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To illustrate the use of the expansion $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$, the problem of a ring with radius a, charge Q and center at $\hat{\mathbf{z}}b$ has been discussed (see page 103 of textbook). By direct expansion of the exact potential $\Phi(z)$ along the z-axis written as a closed square-root expression, we have obtained the terms $l = 0, 1, 2$ in the exterior case $z > c$ and seen that the resultant power series has coefficients involving $P_l(\cos\alpha)$. It was then shown that an application of the expansion $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$ for the case $\mathbf{x} = \hat{\mathbf{z}}z$ and $\mathbf{x}' = \hat{\rho}a + \hat{\mathbf{z}}b$ immediately leads to the expansion of the potential up to arbitrary l, valid for arbitrary $\mathbf{x}$.

Cones and tips with azimuthal symmetry. It was outlined how to solve the Legendre differential equation for Dirichlet BC $P(\cos\beta) = 0$, with $0 < \beta < \pi$. The solutions $P_\nu(\cos\theta)$ are called the Legendre functions of the 1st kind of (generally non-integer) order ν, and they are equivalent to hypergeometric functions ${}_2F_1(-\nu; -\nu+1; 1; \cos\theta)$. Some properties of the $P_\nu(\cos\theta)$ were pointed out. For given β, a countable set of such functions $\{P_{\nu_k}(\cos\theta) | k = 1, 2, ...\}$ satisfies the BC $P_\nu(\cos\beta) = 0$. The significance of the counting index k is that $P_{\nu_k}(\cos\theta)$ has its k-th zero at $\cos\beta$. The set $\{P_{\nu_k}(\cos\theta) | k = 1, 2, ...\}$ is a complete and orthogonal set of functions on the interval $[\cos\beta, 1]$ (with $f(\cos\beta) = 0$). Thus, the general potential near tips and cones is of the form

$$\Phi(\mathbf{x}) = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \tag{27}$$

Reading assignment: Properties of the leading term $k = 1$ of that expansion (textbook p. 106f). Criticality of the lowest order $\nu_1(\beta)$. Qualitative change of field behavior at $\beta = \pi/2$.

Potential expansion in spherical coordinates - general case without azimuthal symmetry.

$$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-1}^{l} \left(A_{lm} r^l + B_{lm} r^{-l-1} \right) Y_{lm}(\theta, \phi) \tag{28}$$

Some properties of the spherical harmonics $Y_{lm}(\theta, \phi)$ were reviewed, including orthogonality and closure relations. The method of how to determine the expansion coefficients in the case of Dirichlet BC on a sphere with radius $r = a$ was briefly reviewed.

Reading assignment: Chapter 3.5 of the textbook (properties of spherical harmonics).

The following important **expansion of the free-space Green's function** has been pointed out:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) = \sum_{l=0}^{\infty} \sum_{m=-1}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi') \tag{29}$$

The **separation of the Laplace equation in cylindrical coordinates** was explained. Two cases of physical significance exist:

Case 1: We consider the case that the potential is zero on a cylinder mantle of radius a. Then, the solutions are of the form $\Phi(\rho,z,\phi) = \exp(\pm i\nu\phi)\exp(\pm kz)\Omega_\nu(k\rho)$, and linear combinations thereof. The index ν usually is $0,1,2...$, and the variable k is real and ≥ 0. The radial function $\Omega_\nu(k\rho)$ is a Bessel function (a solution of the Bessel differential equation).

The properties - asymptotic behavior, roots, etc. - of the Bessel functions $J_\nu(x)$, $J_{-\nu}(x)$, the Neumann functions $N_\nu(x)$, and the Hankel functions $H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$ and $H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$ were discussed. The pair $\{J_\nu(x), N_\nu(x)\}$ is always independent, and so is $\left\{H_\nu^{(1)}(x), H_\nu^{(2)}(x)\right\}$. Most importantly, the function set

$$\left\{\sqrt{\rho}\, J_\nu(\rho\frac{x_{\nu n}}{a}) \,|\, \nu \geq -1 \,\text{and fixed, and}\, n = 1,2,3...\right\} \tag{30}$$

is a complete orthogonal set on the interval $[0,a]$ with BC $f(\rho) = 0$ at $\rho = a$. Thereby, $x_{\nu n}$ stands for the n-th root of $J_\nu(x) = 0$. Note that each value of ν generates a complete orthogonal set. Using the orthogonality relation

$$\int_0^a \rho J_\nu(\rho\frac{x_{\nu n}}{a}) J_\nu(\rho\frac{x_{\nu n'}}{a}) d\rho = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})\delta_{n,n'} \tag{31}$$

any function $f(\rho)$ that vanishes at $\rho = a$ can be expanded as $f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu(\rho\frac{x_{\nu n}}{a})$ with coefficients

$$A_{\nu n} = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu(\rho\frac{x_{\nu n}}{a}) d\rho \tag{32}$$

These findings can be used to expand the potential in cylindrical volumes of radius a with $V = 0$ on the cylinder mantle. If the cylinder is closed on both top and bottom, we can also - by the superposition principle - require that only the top or the bottom lid is on a non-zero potential. To be specific, assume a bottom lid with $V = 0$ at $z = 0$ and a top lid at $z = L$ with potential $V(\rho,\phi)$. Convince yourself by a calculation that the potential in the cylinder can be expanded as

$$\Phi(\mathbf{x}) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z) \left(A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right) \tag{33}$$

with $k_{mn} = \frac{x_{mn}}{a}$ and

$$
\begin{aligned}
A_{mn} &= \frac{2}{\pi a^2 J_{m+1}(k_{mn}a)\sinh(k_{mn}L)} \int_0^{2\pi}\int_0^a \rho V(\rho,\phi) J_m(k_{mn}\rho)\sin(m\phi)d\rho \quad , \quad n=1,2,... \\
B_{mn} &= \frac{2}{\pi a^2 J_{m+1}(k_{mn}a)\sinh(k_{mn}L)} \int_0^{2\pi}\int_0^a \rho V(\rho,\phi) J_m(k_{mn}\rho)\cos(m\phi)d\rho \times \left\{ \begin{array}{ll} 1 & , \quad n=1,2,... \\ 1/2 & , \quad n=0 \end{array} \right.
\end{aligned}
\tag{34}
$$

Case 2: The other case of significance is that the potential is zero on the top and the bottom of the cylinder volume, while on the mantle of radius a the potential is non-zero and equal to $V(z,\phi)$. Assuming that the bottom of the cylinder is located in the plane $z=0$ and the top in the plane $z=L$, the interior solutions are then of the form $\exp(\pm \mathrm{i}\nu\phi)\sin(kz)\Omega_\nu(k\rho)$. There, $\Omega_\nu(k\rho)$ is a modified Bessel function, and the value of k satisfies a quantization condition $ka = n\pi$ with integer n.

The commonly used modified Bessel functions are $I_\nu(x) = \mathrm{i}^{-\nu}J_\nu(\mathrm{i}x)$ and $K_\nu(x) = \frac{\pi}{2}\mathrm{i}^{\nu+1}H_\nu^{(1)}(\mathrm{i}x)$. They are both real and linearly independent. $I_\nu(x)$ is regular at $x=0$ and diverging for $x\to\infty$, while $K_\nu(x)$ is regular for $x\to\infty$, and diverging for $x\to 0$.

If there is **no radial restriction** to the volume of interest, expansions of the potential in non-countable function sets characterized by one or more continuous indices are useful. The potential can, for instance, be expanded into a Fourier integral.

In cases of cylindrically symmetric BC without radial restriction we can use the complete orthogonal set of functions $\left\{\sqrt{\rho}J_m(k\rho) \,|\, k \,\text{real and}\, k\geq 0\right\}$, which has the orthogonality relation

$$
\int_0^\infty \rho J_m(k\rho)J_m(k'\rho)d\rho = \frac{1}{k}\delta(k-k') \qquad \forall\, m \quad . \tag{35}
$$

Thus, in the (charge-free) volume $z>0$ and boundary conditions $\Phi(\rho, z=0, \phi) = V(\rho,\phi)$ the potential is given by a Fourier-Bessel integral

$$
\Phi(\rho,z,\phi) = \sum_{m=0}^\infty \int_0^\infty dk\, \exp(-kz)\, J_m(k\rho)\, \left(A_m(k)\sin(m\phi) + B_m\cos(m\phi)\right) \quad , \tag{36}
$$

where the coefficient functions are

$$
\begin{aligned}
A_m(k) &= \frac{k}{\pi}\int_{\rho=0}^\infty\int_{\phi=0}^{2\pi} \rho V(\rho,\phi)J_m(k\rho)\sin(m\phi)d\rho d\phi \quad , \quad m=1,2,... \\
B_m(k) &= \frac{k}{\pi}\int_{\rho=0}^\infty\int_{\phi=0}^{2\pi} \rho V(\rho,\phi)J_m(k\rho)\cos(m\phi)d\rho d\phi \times \left\{ \begin{array}{ll} 1 & , \quad m=1,2,... \\ 1/2 & , \quad m=0 \end{array} \right.
\end{aligned}
\tag{37}
$$

Reading: Similar methods involving spherical Bessel functions (p119 of textbook).

Good exercise: Derive completeness relations for complete orthogonal sets (CONS) of Bessel functions on a finite interval $[0,a]$, on $[0,\infty[$, and for CONS of spherical Bessel functions on $[0,\infty[$.

Expansion of Green's functions using complete sets of orthogonal functions. The general usefulness of the Green's function has already been discussed earlier. The purpose of the following is to extend the application of expansion methods from mere solutions of the Laplace equation to the Green's function.

Previous example: The free-space Green's function can be expanded in spherical harmonics as

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-1}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \quad . \tag{38}$$

This result is important in the multipole expansion (Chapter 4 of the textbook). It can also be used to expand the Green's functions of problems that can be treated with the image charge method.

Example: Dirichlet Green's function for the exterior volume of a sphere with radius a. For the term in the Green's function that corresponds to the image charge we use the following:

- Relative image charge size $= -\frac{a}{r'}$ with r' being the radial coordinate of the source location $\mathbf{x}'$.
- $r_> = r$, where r is the radial coordinate of the observation coordinate $\mathbf{x}$.
- $r_< = \frac{a^2}{r'}$

Inserting these figures it is found:

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-1}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\,(\frac{a^2}{rr'})^{l+1} \right) \quad . \tag{39}$$

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Systematic methods of how to expand Green's functions. As an example, we consider the Green's function of the volume between two concentric shells with radii $a < b$. This example is lengthier than most homework or exam problems of this kind.

Step 1: Write down the differential equation for the Green's function with δ-function in spherical coordinates,

$$\Delta G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') = -4\pi\frac{\delta(r-r')}{r'^2}\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

Step 2: On right side, use completeness relations for **two out of the three involved δ-functions**. In the present case of spherical coordinates, we only have completeness relations for the angular coordinates. Therefore, we write

$$\Delta G(\mathbf{x},\mathbf{x}') = -4\pi\frac{\delta(r-r')}{r'^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

and re-sort:

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left\{-4\pi\frac{\delta(r-r')}{r'^2}Y_{lm}^*(\theta',\phi')\right\}Y_{lm}(\theta,\phi) \tag{40}$$

Note: In cylindrical or cartesian coordinates, completeness relations are often known for all three of the involved δ-functions, and a choice must be made. Depending on that choice, different but equally valid expansions of the Green's function are obtained (see homework problems).

Step 3: On left side, expand the Green's function using the orthogonal function sets that have also been used in Step 2. Note that $\mathbf{x}'$ only enters as a parameter of the calculation; Δ acts on $\mathbf{x}$.

$$\Delta G(\mathbf{x},\mathbf{x}') = \Delta\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}(r|r',\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Step 4: Write Laplacian in the proper coordinates and take derivatives of **all** orthogonal functions that have been introduced in Steps 2 and 3. It many cases you will have to use a known differential equation to take these derivatives. Typically, you will have to use the (plain or generalized) Legendre differential equation to eliminate angular derivatives of spherical harmonics or Legendre functions. To eliminate radial derivatives of Bessel functions, employ the Bessel differential equation. In the present example, we use the generalized Legendre differential equation.

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]A_{lm}(r|r',\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Using $x=\cos\theta$, it is

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} - \frac{m^2}{r^2\sin^2\theta}\right]A_{lm}(r|r',\theta',\phi')\,Y_{lm}(\theta,\phi)$$

The generalized Legendre differential equation allows us to write $\frac{d}{dx}(1-x^2)\frac{d}{dx}P_l^m(x) = \left[-l(l+1)+\frac{m^2}{1-x^2}\right]P_l^m(x)$, and also $\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}Y_{lm} = \left[-l(l+1)+\frac{m^2}{1-x^2}\right]Y_{lm}$. Thus, with $1-x^2=\sin^2\theta$

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1)}{r^2} + \frac{m^2}{r^2\sin^2\theta} - \frac{m^2}{r^2\sin^2\theta}\right]A_{lm}(r|r',\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left\{\left[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}\right]A_{lm}(r|r',\theta',\phi')\right\}Y_{lm}(\theta,\phi) \tag{41}$$

Step 5: Expansions in orthogonal sets of functions are unique. Thus, we can separately equate the coefficients of the Y_{lm} in Eq. 40 and Eq. 41. Dividing the resultant equation by $Y_{lm}^*(\theta',\phi')$, we find

$$\left[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}\right]\left\{\frac{A_{lm}(r|r',\theta',\phi')}{Y_{lm}^*(\theta',\phi')}\right\} = -4\pi\frac{\delta(r-r')}{r'^2}$$

Step 6: Noticing that the expression in the curly brackets of the last equation can possibly only depend on l, r and r', we define the reduced (radial) Green's function

$$g_l(r,r') = \frac{A_{lm}(r|r',\theta',\phi')}{Y^*_{lm}(\theta',\phi')} \quad .$$

Note that the radial Green's function only depends on one of the two indices of the orthogonal function set employed in Steps 2 and 3. This is an exception. In most other cases, reduced Green's functions will depend on all indices of the utilized orthogonal function sets.

We proceed to solve

$$\left[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}\right] g_l(r,r') = -4\pi\frac{\delta(r-r')}{r'^2} \tag{42}$$

Outside the location of the δ-function inhomogeneity, the solutions are of the form $g_l(r,r') = Ar^l - Br^{-l-1}$. More specifically, to match the boundary condition $g_l(r=a,r')=0$, for $r<r'$ it must be $g_l(r,r') \propto r^l - \frac{a^{2l+1}}{r^{l+1}}$. To match the boundary condition $g_l(r=b,r')=0$, for $r>r'$ it must be $g_l(r,r') \propto r^{-l-1} - \frac{r^l}{b^{2l+1}}$. Further, $g_l(r,r')$ must be symmetric in r and r', and it must be continuous at $r=r'$. To satisfy all these conditions, $g_l(r,r')$ must be of the form

$$g_l(r,r') = C\left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(r^{-l-1}_> - \frac{r^l_>}{b^{2l+1}}\right) \quad ,$$

with a constant C to be determined to match the behavior near the δ-function inhomogeneity. Also, $r_< = \min(r,r')$ and $r_> = \max(r,r')$.

Step 7: Find C. Integrating Eq. 42 from $r'-\epsilon$ to $r'+\epsilon$ with $\epsilon \to 0$ and dropping vanishing terms we find

$$\frac{d}{dr}(rg_l(r,r'))|_{r=r'+\epsilon} - \frac{d}{dr}(rg_l(r,r'))|_{r=r'-\epsilon} = -\frac{4\pi}{r'}$$

A lengthy but simple calculation yields

$$C = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}$$

Step 8:

Going backward through all steps, the Green's function expansion is assembled into the final result:

$$g_l(r,r') = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(r^{-l-1}_> - \frac{r^l_>}{b^{2l+1}}\right) \quad ,$$

$$A_{lm}(r|r',\theta',\phi') = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(r^{-l-1}_> - \frac{r^l_>}{b^{2l+1}}\right) Y^*_{lm}(\theta',\phi')$$

and finally

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Reading: Use of the above Green's function to calculate the potential produced by charge distributions (Chapter 3.10 of the textbook).

Reading: Application of the expansion method to expand the free-space Green's function in cylindrical coordinates (Chapter 3.11 of the textbook).

Eigenfunction expansion of the Green's function.

General method. The solutions $\psi(\mathbf{x})$ of the eigenvalue problem

$$(\Delta + f(\mathbf{x}) + \lambda)\psi(\mathbf{x}) = 0$$

with eigenvalues λ can be arranged to form an orthonormal, complete set of functions on the definition rage of the equation. Boundary conditions may apply. Closed volumes lead to countable sets $\{\psi_n(\mathbf{x}), n = 1, 2, ..\}$ with corresponding eigenvalues λ_n, while open volumes have sets with at least one continuous parameter. In free space and $f(\mathbf{x}) = 0$, for instance, one may choose $\left\{\psi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi^3}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}), \mathbf{k} \in \mathbb{R}^3\right\}$ with eigenvalues $\lambda_{\mathbf{k}} = k^2$.

The analogy of the situation to quantum mechanics has been discussed. In that language, $\Delta + f(\mathbf{x})$ is a Hermitian linear differential operator. As is well known, such operators plus their boundary conditions generate orthonormal, complete sets of eigenfunctions.

It has been shown that orthogonality and completeness of the set $\{\psi_n(\mathbf{x})\}$ with eigenvalues $\{\lambda_n\}$ lead to an expansion of the Green's function. Assume

$$(\Delta + f(\mathbf{x}) + \lambda_n)\psi_n(\mathbf{x}) = 0 \quad ,$$

with $\psi_n(\mathbf{x})$ satisfying any boundary conditions that may apply. The Green's function is defined as the solution of

$$(\Delta + f(\mathbf{x}) + \lambda)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \quad ,$$

with $G(\mathbf{x},\mathbf{x}')$ satisfying the same boundary conditions as the $\psi_n(\mathbf{x})$. It follows

$$G_\lambda(\mathbf{x},\mathbf{x}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{x}')\,\psi_n(\mathbf{x})}{\lambda_n - \lambda} \quad .$$

In the special **context of the Laplace equation**, set $f(\mathbf{x}) = 0$ and $\lambda = 0$ in the above equations. Then, for discrete sets of eigenfunctions satisfying $(\Delta + \lambda_n)\psi_n(\mathbf{x}) = 0$, $n = 1, 2, ..$, boundary conditions, and orthonormality $\int \psi_n^*(\mathbf{x})\,\psi_{n'}(\mathbf{x}) d^3x = \delta_{n,n'}$ we have

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{x}')\,\psi_n(\mathbf{x})}{\lambda_n} \quad .$$

For sets of eigenfunctions with continuous eigenvalues satisfying $(\Delta + \lambda_k)\psi_k(\mathbf{x}) = 0$, continuous k , boundary conditions (if any), and orthonormality $\int \psi_k^*(\mathbf{x})\,\psi_{k'}(\mathbf{x})d^3x = \delta(k-k')$ we have

$$G(\mathbf{x},\mathbf{x}') = 4\pi \int_k \frac{\psi_k^*(\mathbf{x}')\,\psi_k(\mathbf{x})}{\lambda_k}\,dk \quad .$$

Sometimes there is more than one continuously variable index for the eigenfunctions. Also, combinations of continuous and discrete indices exist. In all these cases, add integrals or sums in the above equations as appropriate.

Two examples have been briefly discussed (rectangular box and free space).

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Multipole expansion. The first case in which such expansions are useful is if we are concerned with the potential generated by a small charge distribution in an otherwise charge- and field-free space. We consider the potential of a charge distribution $\rho(\mathbf{x}')$ of typical extension R at locations $\mathbf{x}$ with $|\mathbf{x}| = r > R$. Then, application of the free-space Green's function expansion Eq. 38 yields

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{1}{r^{l+1}}\, Y_{lm}(\theta,\phi) \tag{43}$$

with **spherical multipole moments**

$$q_{lm} = \int \rho(\mathbf{x}')\, r'^l\, Y_{lm}^*(\theta',\phi')\, d^3x' \tag{44}$$

Alternately, an expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in cartesian coordinates for fixed $\mathbf{x}$ around $\mathbf{x}' = 0$ yields

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left\{ \sum_{\substack{k,m,n=0 \\ k+m+n=l}}^{l} \left[\frac{1}{k!m!n!} \frac{\partial^k}{\partial x'^k} \frac{\partial^m}{\partial y'^m} \frac{\partial^n}{\partial z'^n} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \right]_{\mathbf{x}'=0} \int \rho(\mathbf{x}') x'^k y'^m z'^n\, dx'dy'dz' \right\} \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left\{ \sum_{\substack{k,m,n=0 \\ k+m+n=l}}^{l} \frac{P_{kmn}^{(l)}(x,y,z,r)}{r^{2l+1}} Q_{kmn}^{(l)} \right\}
\end{aligned} \tag{45}$$

Thereby, l identifies the order of the expansion terms (as in the expansion in spherical coordinates). $P^{(l)}_{kmn}$ is a l-th order polynomial in the observation point coordinates $(x, y, z, r = \sqrt{x^2 + y^2 + z^2})$, leading to an overall radial dependence of the l-th order terms $\propto \frac{1}{r^{l+1}}$ (as in the expansion in spherical coordinates). Further, $Q^{(l)}_{kmn}$ is the l-th cartesian multipole moment

$$Q^{(l)}_{kmn} = \int \rho(\mathbf{x}') x'^k y'^m z'^n \, dx' dy' dz' \tag{46}$$

The monopole ($l = 0$) and dipole ($l = 1$) terms have been stated. It has been shown in class that after resorting of terms the quadrupole ($l = 2$) contribution takes the form given in Eq. 4.10 of the textbook, with the quadrupole tensor given in Eq. 4.9. Unless noted otherwise, Eq. 4.9 and 4.10 are used to calculate cartesian quadrupole moments and their potentials and fields.

To further demonstrate the systematics in the cartesian multipole expansion, in the following the next terms are given. You may check that the term $l = 3$ is

$$\begin{aligned}
\Phi_3(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \sum_{\substack{i,j,k=1 \\ i \le j \le k}}^{3} \left[\frac{1}{A_{ijk}} \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x'_j} \frac{\partial}{\partial x'_k} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right]_{\mathbf{x}'=0} \int \rho(\mathbf{x}') x'_i x'_j x'_k \, d^3x' \\
&= \frac{1}{4\pi\epsilon_0} \sum_{\substack{i,j,k=1 \\ i \le j \le k}}^{3} \frac{15 x_i x_j x_k - 3r^2(\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j)}{A_{ijk} \, r^7} \tilde{Q}_{ijk}
\end{aligned}$$

where the indices i, j, k refer to the three components of cartesian coordinates, $\tilde{Q}_{ijk}$ is the octupole moment $\tilde{Q}_{ijk} = \int \rho(\mathbf{x}') x'_i x'_j x'_k \, d^3x'$, and $A_{ijk} = 1$ if all indices are different, $A_{ijk} = 2$ if exactly two indices are equal and $A_{ijk} = 6$ if all indices are equal. The general form of this result conforms with the $l = 3$-term of Eq. 45; note, however, the different meanings of the indices. It is, for instance, $\tilde{Q}_{122} = Q^{(3)}_{120}$.

Similarly, you should find for $l = 4$

$$\begin{aligned}
\Phi_4(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \sum_{\substack{i,j,k,m=1 \\ i \le j \le k \le m}}^{3} \left[\frac{1}{A_{ijkm}} \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x'_j} \frac{\partial}{\partial x'_k} \frac{\partial}{\partial x'_m} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right]_{\mathbf{x}'=0} \int \rho(\mathbf{x}') x'_i x'_j x'_k x'_m \, d^3x' \\
&= \frac{1}{4\pi\epsilon_0} \sum_{\substack{i,j,k,m=1 \\ i \le j \le k \le m}}^{3} \frac{H_{ijkm}(\mathbf{x})}{A_{hijk} \, r^9} \tilde{Q}_{ijkm} \quad \text{with} \\
H_{ijkm}(\mathbf{x}) &= 105 x_i x_j x_k x_m - 15r^2(\delta_{ij} x_k x_m + \delta_{jk} x_i x_m + \delta_{ik} x_j x_m + \delta_{im} x_j x_k + \delta_{jm} x_i x_k + \delta_{km} x_i x_j) \\
&+ 3r^4(\delta_{ij}\delta_{km} + \delta_{jk}\delta_{im} + \delta_{ik}\delta_{jm})
\end{aligned} \tag{47}$$

There, $\tilde{Q}_{ijkm} = \int \rho(\mathbf{x}') x'_i x'_j x'_k x'_m \, d^3x'$, and $A_{ijkm} = 2$ if exactly two indices are equal, $A_{ijkm} = 6$ if exactly three indices are equal, and $A_{ijkm} = 24$ if all indices are equal. This result is of the same form

as the $l = 4$-term in Eq. 45. Again, note the different meanings of the indices. It is, for instance, $\tilde{Q}_{1123} = Q^{(4)}_{211}$.

Important fact a: Out of the $(l+2)(l+1)/2$ cartesian multipole moments $Q^{(l)}_{kmn}$ of order l, defined in Eq. 46, there are only $2l+1$ independent ones. This number equals the number of spherical moments of order l; the latter are all independent from each other.

Important fact b: The lowest-order non-vanishing multipole moments are invariant under translations of the origin (not under rotations).

For the cartesian moments, this fact follows from a transformation rule obtained in the following. Assume that the multipole moments in a certain frame are labeled $Q^{(l)}_{kmn}$. The origin is then translated by $\mathbf{x}_0 = (x_0, y_0, z_0)$. Then, the transformed moments $\hat{Q}^{(l)}_{kmn}$ are

$$\begin{aligned}
\hat{Q}^{(l)}_{kmn} &= \int \rho(\mathbf{x})(x-x_0)^k (y-y_0)^m (z-z_0)^n \, dxdydz \\
&= \sum_{a=0}^{k}\sum_{b=0}^{m}\sum_{c=0}^{n} \left(\begin{array}{c} k \\ a \end{array}\right)\left(\begin{array}{c} m \\ b \end{array}\right)\left(\begin{array}{c} n \\ c \end{array}\right) (-x_0)^{k-a} (-y_0)^{m-b} (-z_0)^{n-c} \int \rho(\mathbf{x}) x^a y^b z^c \, dxdydz \\
&= \sum_{a=0}^{k}\sum_{b=0}^{m}\sum_{c=0}^{n} \left(\begin{array}{c} k \\ a \end{array}\right)\left(\begin{array}{c} m \\ b \end{array}\right)\left(\begin{array}{c} n \\ c \end{array}\right) (-x_0)^{k-a} (-y_0)^{m-b} (-z_0)^{n-c} Q^{(a+b+c)}_{abc} \\
&= \left\{ \sum_{a=0}^{k}\sum_{b=0}^{m}\sum_{c=0}^{n} \Big|_{a+b+c<l} \left(\begin{array}{c} k \\ a \end{array}\right)\left(\begin{array}{c} m \\ b \end{array}\right)\left(\begin{array}{c} n \\ c \end{array}\right) (-x_0)^{k-a} (-y_0)^{m-b} (-z_0)^{n-c} Q^{(a+b+c)}_{abc} \right\} + Q^{(l)}_{kmn}
\end{aligned} \tag{48}$$

There, the $(*)$ are binomial coefficients. Since the expression in the curly brackets only contains multipole moments of orders lower than l, it is shown that the lowest-order non-vanishing multipole moments are independent of translations of the origin.

Important fact c: The contributions of the multipoles to the potential scale as $\sim \frac{1}{r}\left(\frac{R}{r}\right)^l$, where R is the size of the source and the origin is assumed to be chosen inside the source (which is a prerequisite for the expansion to make sense). This scaling behavior explains why the multipole expansion is a powerful tool.

Review the following:

- Electric fields associated with multipole moments.
- Potentials and fields of an electric dipole.
- Distinction and relation between idealized multipoles and their approximate realizations. The matter has been discussed in class for an electric dipole.

In preparation of future subjects, the following results have been obtained. Consider the integral over the electric field inside a sphere with radius R. If the sphere **includes all charges** $\rho(\mathbf{x})$, then

$$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{\mathbf{p}}{3\epsilon_0} \quad \text{with dipole moment} \quad \mathbf{p} = \int_{r<R} \mathbf{x}\rho(\mathbf{x}) d^3x \quad . \tag{49}$$

If all charges are outside the sphere, then the average electric field in the sphere equals the field at the center, *i.e.*

$$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = \frac{4\pi}{3} R^3 \, \mathbf{E}(\mathbf{0}) \quad . \tag{50}$$

Reading: detailed derivation of these equations in Chapter 4.1.

The second case in which multipole expansions are useful is if one is concerned with the energy of a charge distribution $\rho(\mathbf{x})$ of typical size R in an external electric field that varies over length scales much larger than R. We assume that the shape of the distribution $\rho(\mathbf{x})$ is fixed, while its position and orientation with respect to the external sources may be varied. The situation is analyzed in a body frame of the charge distribution; the origin of that frame should be chosen inside the distribution. Expansion of the external potential around the origin of the body frame yields

$$\begin{aligned} W &= q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j}{\partial x_j}(0) + ... \quad \text{with} \\ q &= \int \rho(\mathbf{x})\, d^3x \\ \mathbf{p} &= \int \mathbf{x}\,\rho(\mathbf{x})\, d^3x \\ Q_{ij} &= \int (3x_i x_j - r^2\delta_{ij})\,\rho(\mathbf{x})\, d^3x \end{aligned} \tag{51}$$

There, $\Phi(0)$, $\mathbf{E}(0)$ and $\frac{\partial E_j}{\partial x_j}(0)$ are the potential, the field and the field derivatives due to the external sources at the origin (which is fixed with respect to the charge distribution). The self-energy of the charge distribution is **not** included in this result. The energy W depends on the location and the orientation of the body and its charge distribution relative to the external field sources.

The following applications of Eq. 51 have been discussed:

- Interaction energy between two electric dipoles. See Eq. 4.26 in textbook.
- Hyperfine structure (HFS) of atoms. The HFS is due to the interactions between the multipoles of the nuclear charge distribution with the field produced by the electron system at the nucleus. The magnetic-dipole interaction, which presently is not of interest, has been pointed out because it normally dominates the HFS. Electric-dipole and magnetic-quadrupole interactions between nuclei and electron shells are identical zero due to the well-defined parity of the nuclear wavefunction. Therefore, the next-higher HFS term, which we are concerned with in the present context, is the electric-quadrupole interaction. The Casimir formula, derived 1936, was quoted (H. B. G.

Casimir, *On the Interaction Between Atomic Nuclei and Electrons*, reprinted by W. H. Freeman, San Francisco (1963)):

$$\begin{aligned} W_{HFS,E2} &= -\frac{1}{4}\left.\frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0} \mathrm{e}Q_0 \, \frac{\frac{3}{2}C(C+1) - 2I(I+1)J(J+1)}{I(2I-1)J(2J-1)} \quad \text{with} \\ C &= F(F+1) - I(I+1) - J(J+1) \quad . \end{aligned} \tag{52}$$

There, I, J, and F are the nuclear spin, total electronic spin (orbital plus intrinsic), and hyperfine quantum numbers, respectively. Further, $\left.\frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0}$ is the z-derivative of the z-electric field produced by the electron system at the location of the nucleus, whereby the z-axis is taken parallel to the total electronic spin $\mathbf{J}$. Finally, Q_0 is the nuclear quadrupole moment $Q_0 = \frac{1}{e}\int (3z'^2 - r'^2)\rho(\mathbf{x}')\, d^3x'$, calculated in a body frame the z-axis of which coincides with the nuclear symmetry axis (if one models the nucleus as a classical charge distribution). The homework problem 4.6 corresponds to the case $F = I + J$.
Note. To solve the homework problem, the above details are not required. You only need the cylindrical symmetry of the electric field and of the nuclear charge distribution about the z-axis.

Electrostatics in dielectric media. In dielectric media, it is desirable to have differential equations for a macroscopic electric field. The macroscopic field is the volume average of the microscopic field, $\mathbf{E}(\mathbf{x}) = \langle \mathbf{E}_{\mathrm{micro}}(\mathbf{x}) \rangle_{\mathrm{volume}}$. The averaging volume is small on a macroscopic scale, but contains many molecules. In contrast to the microscopic field, which is complicated and normally unnecessary to know, the macroscopic field follows simple equations. The microscopically valid (and therefore always valid) homogeneous equation $\nabla \times \mathbf{E}_{\mathrm{micro}} = 0$ can be averaged: $0 = \langle \nabla \times \mathbf{E}_{\mathrm{micro}} \rangle_{\mathrm{volume}} = \nabla \times \langle \mathbf{E}_{\mathrm{micro}} \rangle_{\mathrm{volume}} = \nabla \times \mathbf{E}$. Thus, it is

$$\nabla \times \mathbf{E} = 0 \quad ,$$

which is the same equation as in free space. We conclude that in dielectric media the electric field will still be derivable from an electric potential $\Phi(\mathbf{x})$.

The inhomogeneous equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ is modified as follows. The dielectric medium is assumed to contain molecules of species i with corresponding volume densities N_i. One molecule of species i carries an average charge q_i (which is usually zero) and an average molecular dipole $\mathbf{p}_i$. Higher-order multipoles of the molecules are neglected (which is an exceedingly good approximation). We account for free charges (and molecular charges, if $\neq 0$) in the free-charge density $\rho_{\mathrm{free}}(\mathbf{x})$; the subscript is usually omitted and we just write $\rho(\mathbf{x})$. The molecular dipole moments are accounted for via a macroscopic polarization $\mathbf{P}(\mathbf{x}) = \sum_i N_i(\mathbf{x})\mathbf{p}_i(\mathbf{x})$. It was then derived in class that

$$\nabla \cdot [\epsilon_0 \mathbf{E}(\mathbf{x}) + \mathbf{P}(\mathbf{x})] = \nabla \cdot \mathbf{D}(\mathbf{x}) = \rho(\mathbf{x}) \quad ,$$

where $[\epsilon_0 \mathbf{E} + \mathbf{P}] = \mathbf{D}$ defines the **electric displacement field D**.

Once the polarization $\mathbf{P}(\mathbf{x})$ is known, the **volume polarization charge** $\rho_{\mathrm{pol}}(\mathbf{x}) = -\nabla \cdot \mathbf{P}(\mathbf{x})$ can be calculated. On boundaries of polarized media to the vacuum, there is a **surface polarization charge** $\sigma_{\mathrm{pol}}(\mathbf{x}) = \hat{\mathbf{n}} \cdot \mathbf{P}(\mathbf{x})$, where the normal vector $\hat{\mathbf{n}}$ is pointing from the medium outward. These polarization charges are **not included** in the density of free charges $\rho(\mathbf{x})$. In contrast, the variable ρ_{all} in the equation for the microscopic field, $\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho_{\mathrm{all}}}{\epsilon_0}$, includes both free and polarization charge. Thus, questions

about charge densities in dielectric media have to be handled very carefully, and a clear distinction between free and polarization charges must be made.

Reading and review: Definition of linear and isotropic dielectric media. Definitions of electric permittivity, dielectric constant and electric susceptibility.

Boundary conditions at interfaces between dielectric media. Assume fields $\mathbf{E}_1$ and $\mathbf{D}_1$ at the boundary of region 1 and fields $\mathbf{E}_2$ and $\mathbf{D}_2$ at the boundary of region 2. Then,

$$\begin{aligned} [\mathbf{D}_2(\mathbf{x}) - \mathbf{D}_1(\mathbf{x})] \cdot \hat{\mathbf{n}} &= \sigma(\mathbf{x}) \\ [\mathbf{E}_2(\mathbf{x}) - \mathbf{E}_1(\mathbf{x})] \times \hat{\mathbf{n}} &= 0 \quad , \end{aligned} \tag{53}$$

with the normal vector $\hat{\mathbf{n}}$ pointing from region 1 to region 2. Thus, the component of $\mathbf{D}$ normal to the interface displays a discontinuity of size σ - the surface density of **free** charges -, while the components of $\mathbf{E}$ in plane with the interface are continuous. The validity of these boundary conditions does not require linear and / or isotropic behavior of the dielectric.

Examples discussed in class:

- In infinite volumes with constant permittivity ϵ, the electric field follows the equations $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\epsilon}$. Thus, in electrostatic equations containing ϵ_0 - such as the potentials and fields due to localized charges, the potentials and fields due to charge distributions, capacitances, electrostatic energy, energy of capacitors - the dielectric medium is accounted for by replacing ϵ_0 by the permittivity $\epsilon = \epsilon_0 \epsilon_r$ (where ϵ_r is the dielectric constant).

- The polarization charge at a dielectric interface with given $\mathbf{E}$- and $\mathbf{D}$-fields on both sides and with zero free charges has been calculated.

- The potential $\Phi(\mathbf{x})$ due to a polarized object with given polarization $P(\mathbf{x})$ in an otherwise source- and field-free volume can be obtained by calculation of the polarization charges and subsequent use of a microscopic equation for $\Phi(\mathbf{x})$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left(\int_{V \setminus \partial V} \frac{-\nabla' \cdot P(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x' + \int_{\partial V} \frac{\hat{\mathbf{n}}' \cdot P(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, da' \right) \quad , \tag{54}$$

  where the volume integral is only over the interior of the polarized object. That is, don't extend the volume integral over the discontinuity of $P(\mathbf{x})$, because that would amount to double-counting the effect of the surface polarization charge.

- The solution of the following image charge problem has been outlined: The volume $z > 0$ has permittivity ϵ_1, and the volume $z < 0$ has permittivity ϵ_2. A (free) point charge q is located at $(0, 0, d)$ with $d > 0$. The problem can be solved by assuming two image charges at locations $(0, 0, \pm d)$ and consideration of the boundary conditions for D_z and for E_x, E_y on the interface plane $z = 0$.
  Reading: Details of this example in Chapter 4.4 of the textbook.

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The image charge problem from the previous lecture was used in order to demonstrate how volume and polarization charges are obtained once the electric potential $\Phi(\mathbf{x})$ is established:

1) Obtain the polarization $\mathbf{P}(\mathbf{x}) = -(\epsilon_i - \epsilon_0)\nabla\Phi(\mathbf{x})$ for the involved regions with permittivities ϵ_i.

2) The polarization volume charge is then $\rho_{\rm pol} = -\nabla \cdot \mathbf{P}(\mathbf{x})$.

3) The surface polarization charge at interfaces between regions labeled 1 and 2 is $\sigma_{\rm pol} = \hat{\mathbf{n}} \cdot (\mathbf{P}_1 - \mathbf{P}_2)$, where $\hat{\mathbf{n}}$ is the normal vector pointing from region 1 to region 2.

3) Via integration of result 2) over a small volume including a point charge and use of Eq. 2, or via a consideration based on Gauss's law in integral form, it is seen that point charges of size q are surrounded by a $\delta-$function polarization charge $q_{\rm pol} = q(\frac{\epsilon_0}{\epsilon} - 1)$, where ϵ is the permittivity of the medium.

Other example. These concepts have been further explained using the example of a dielectric sphere with radius a and permittivity ϵ placed in an initially homogeneous electric field $\hat{\mathbf{z}}E_0$. There, the potential is obtained via variable separation in spherical coordinates:

Inside: $\Phi_{\rm in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

Outside: $\Phi_{\rm in} = \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta) - E_0 r P_1(\cos\theta)$

The boundary condition for $\mathbf{D}$ is $\epsilon \frac{\partial}{\partial r}\Phi_{\rm in}|_{r=a} = \epsilon_0 \frac{\partial}{\partial r}\Phi_{\rm out}|_{r=a}$, and the boundary condition for $\mathbf{E}$ is $\frac{\partial}{r\partial\theta}\Phi_{\rm in}|_{r=a} = \frac{\partial}{r\partial\theta}\Phi_{\rm out}|_{r=a}$. The latter, together with $A_0 = C_0 = 0$, is equivalent with $\Phi_{\rm in}|_{r=a} = \Phi_{\rm out}|_{r=a}$. Equating the coefficients of the Legendre polynomials (or the derivatives of these), an algebraic equation for the coefficients A_l, C_l is found:

$$\begin{pmatrix} \epsilon l a^{l-1} & \epsilon_0 (l+1) a^{-l-2} \\ a^{l-1} & -a^{-l-2} \end{pmatrix} \begin{pmatrix} A_l \\ C_l \end{pmatrix} = \begin{pmatrix} -\epsilon_0 E_0 \delta_{l,1} \\ -E_0 \delta_{l,1} \end{pmatrix}$$

The solution can be found with Kramer's rule, and is given in the textbook (page 158f). The resultant fields and polarization charges have been obtained.

Molecular polarizability. The Clausius-Mossotti equation has been derived. This equation relates the microscopic molecular polarizability $\gamma_{\rm mol}$ with the macroscopic electric permittivity ϵ. There, $\gamma_{\rm mol}$ is defined through the linear relation between microscopic electric dipole moment (per molecule) $\mathbf{p}$ and the microscopic electric field $\mathbf{E}_{\rm mol}$ at the exact location of the molecule, $\mathbf{p} = \epsilon_0 \gamma_{\rm mol} \mathbf{E}_{\rm mol}$. An important finding leading to the **Clausius-Mossotti equation** is a relation between macroscopic (volume-averaged) field $\mathbf{E}$, the macroscopic polarization $\mathbf{P}$ and the microscopic electric field $\mathbf{E}_{\rm mol}$, stating

$$\mathbf{E}_{\rm mol} = \mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0} \quad .$$

This equation is valid for most crystalline and amorphous media, and has been derived in class in three ways. Using this equation, $\mathbf{P} = N\mathbf{p}$ with N denoting the number of molecules per volume, and $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$, the Clausius-Mossotti equation is obtained:

$$\gamma_{\mathrm{mol}} = \frac{3}{N} \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) \quad .$$

The equation is important because it relates the microscopic molecular parameter γ_{mol} with a macroscopic, phenomenological quantity (the dielectric constant $\frac{\epsilon}{\epsilon_0}$).

Microscopic models for the molecular polarizability γ_{mol}, based on classical mechanics and classical statistical mechanics, have been briefly discussed. The distinction between **induced and orientation polarization** has been pointed out.
Reading: Chapter 4.6 of the textbook.

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Energy considerations in dielectric media. Various important equations and their range of applicability have been discussed:

A charge distribution $\rho(\mathbf{x})$ is assumed to generate a potential $\Phi(\mathbf{x})$, and no charges other than those included in $\rho(\mathbf{x})$ are supposed to be present. The positive-definite total electrostatic energy, which can be interpreted as the self-energy of the whole distribution $\rho(\mathbf{x})$, then is

$$W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3x = \frac{1}{2} \int \mathbf{E}(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}) d^3x \quad . \tag{55}$$

This equation is valid in vacuum and in **linear dielectric media**. In the latter case, polarization energy and field energy is included in the total electrostatic energy W, and $\rho(\mathbf{x})$ denotes free charge and excludes polarization charges (as usual).

A more general equation, valid for electrostatic energy in any dielectric, is

$$W = \int_V \left\{ \int_{\mathbf{D}=\mathbf{0}}^{\mathbf{D}} \mathbf{E}\left[\mathbf{D}, \text{ path}\right] \cdot \delta \mathbf{D} \right\} d^3x \quad .$$

This equation also covers cases of nonlinear behavior and/or hysteresis.

Eq. 55 can be used to show the following. The introduction of a **linear dielectric body** from a field-free region into a field region with **initial value** $\mathbf{E}_0(\mathbf{x})$, generated by **fixed sources**, results in a change of electrostatic energy by the amount

$$W = -\frac{1}{2} \int_V \mathbf{P}(\mathbf{x}) \cdot \mathbf{E}_0(\mathbf{x}) d^3x$$

where the integral goes over the volume of the dielectric, and $\mathbf{P}(\mathbf{x})$ denotes its **final-state** polarization. Thus, electrostatic energy is reduced when a dielectric body is brought into a region with increased electric field. The energy balance is converted into translational and/or rotational kinetic energy. Therefore, mechanical forces can be derived from

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q \quad ,$$

where the subscript reminds us that the charges are held fixed (in capacitor problems, this is the case of "disconnected batteries"). The coordinate ξ can be any generalized coordinate of the dielectric body (position, angle, etc.); the force equation can therefore also be used to derive torques, for instance.

A complementary equation for the case of **fixed potentials on the boundaries** exists:

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V \quad ,$$

where the subscript indicates that the boundary potentials are kept fixed. In capacitor problems, this corresponds to the case of "connected batteries".

When a dielectric body moves into a region of increased electric field while the boundary potentials are kept fixed, the kinetic energy of the body and the total electrostatic energy increase by the same amount. Thus, the change in total energy (electrostatic plus kinetic) equals twice the change in electrostatic energy. The energy is provided by the (connected) batteries, which supply the boundary surfaces with charges required to maintain their fixed potentials while the dielectric moves into the field.

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14 10/21/2003 (Review session)

Some specific information on the exam material was provided (see e-mailed and posted exam information).

Elements of the lecture notes were reviewed.

The example of a dielectric sphere with radius a and permittivity ϵ placed in an initially homogeneous electric field $\hat{\mathbf{z}}E_0$ was explained in more detail.

Potential inside: $\Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

Potential outside: $\Phi_{\text{in}} = \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta) - E_0 r P_1(\cos\theta)$

The boundary condition for $\mathbf{D}$ is $\epsilon \frac{\partial}{\partial r}\Phi_{\text{in}}|_{r=a} = \epsilon_0 \frac{\partial}{\partial r}\Phi_{\text{out}}|_{r=a}$, and the boundary condition for $\mathbf{E}$ is $\frac{\partial}{r\partial\theta}\Phi_{\text{in}}|_{r=a} = \frac{\partial}{r\partial\theta}\Phi_{\text{out}}|_{r=a}$. The latter, together with $A_0 = C_0 = 0$, is equivalent with $\Phi_{\text{in}}|_{r=a} = \Phi_{\text{out}}|_{r=a}$. Equating the coefficients of the Legendre polynomials (or the derivatives of these), an algebraic equation for the coefficients A_l, C_l is found:

$$\begin{pmatrix} \epsilon l a^{l-1} & \epsilon_0 (l+1) a^{-l-2} \\ a^{l-1} & -a^{-l-2} \end{pmatrix} \begin{pmatrix} A_l \\ C_l \end{pmatrix} = \begin{pmatrix} -\epsilon_0 E_0 \delta_{l,1} \\ -E_0 \delta_{l,1} \end{pmatrix}$$

Determinant: $D(l) = -(l\epsilon + (l+1)\epsilon_0) a^{-3}$. Since $D \neq 0 \quad \forall \quad l$, it is $A_l = C_l = 0$ unless $l = 1$.

In the case $l = 1$, use Kramer's rule to find

$$A_1 = \frac{3\epsilon_0 E_0 a^{-3}}{-(\epsilon + 2\epsilon_0)a^{-3}} = -E_0 \frac{3}{2 + \epsilon/\epsilon_0} \quad .$$

$$C_1 = \frac{-(\epsilon - \epsilon_0)E_0}{-(\epsilon + 2\epsilon_0)a^{-3}} = E_0 a^3 \frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \quad .$$

The similarity of this example with homework problem 4.8 was pointed out.

In context with electrostatic energy in dielectrics, the following example has been discussed. A planar capacitor with plate separation d, length $L \gg d$ and width $b \gg d$ is initially filled with air ($\epsilon = \epsilon_0$). A dielectric material with permittivity $\epsilon > \epsilon_0$, thickness d, length L and width b is then allowed to move without friction into the capacitor until it fills its entire interior. One capacitor plate is on a fixed potential 0, the other on a fixed potential V.

Force: All fields are perpendicular to the field plates. The electric field is $E = V/d$, the electric displacement $D = \epsilon_0 E$ in the air-filled region and $D = \epsilon E$ in the region filled with the dielectric. Denoting the length of the dielectric that is already inside the capacitor as x, the dielectric energy is

$$W(x) = \frac{1}{2} \int \mathbf{E}(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}) d^3x = \frac{V^2\, b}{2d} \left(\epsilon x + \epsilon_0 (L - x)\right) \quad ,$$

and the force that pulls the dielectric further into the capacitor is

$$F_x = \left(\frac{\partial W}{\partial x}\right)_V = \frac{V^2\, b}{2d} \left(\epsilon - \epsilon_0\right) \quad .$$

Change in electrostatic energy (field plus polarization): As the dielectric moves from entirely outside to entirely inside the capacitor, the electrostatic energy changes by

$$\Delta W_{\rm el} = W(L) - W(0) = \frac{V^2 L\, b}{2d} \left(\epsilon - \epsilon_0\right) = \frac{V^2}{2} \left(C_{\rm f} - C_{\rm i}\right) \quad ,$$

with $C_{\rm f}$ and $C_{\rm i}$ denoting final and initial capacitances.

The **kinetic energy** of the dielectric increases by the same amount,

$$\Delta W_{\rm kin} = F_x L = \frac{V^2 L\, b}{2d} \left(\epsilon - \epsilon_0\right) = \frac{V^2}{2} \left(C_{\rm f} - C_{\rm i}\right) \quad .$$

The **work done by the battery** keeping the capacitor plates on fixed potentials is

$$\Delta W_{\rm bat} = \Delta W_{\rm el} + \Delta W_{\rm kin} = \frac{V^2 L\, b}{d} \left(\epsilon - \epsilon_0\right) = V^2 \left(C_{\rm f} - C_{\rm i}\right) \quad .$$

The **charge densities** on the more positive plate are $\sigma_{\rm free} = \epsilon_0 E$ and $\sigma_{\rm free} = \epsilon E$ in the air-filled and dielectric-filled portions, respectively. The **polarization surface charge density** in the dielectric-filled portion is $\sigma_{\rm pol} = -(\epsilon - \epsilon_0)E$ and zero elsewhere. Reverse all signs to obtain the surface charges on the

more negative plate and the adjacent dielectric surface. The **volume polarization charge density** is zero everywhere.

The amount of **free charge flowing onto the capacitor plates** during the process is $\Delta q = Lb\,\Delta\sigma_{\text{free}} = Lb(\epsilon - \epsilon_0)E = V(\epsilon - \epsilon_0)\frac{Lb}{d} = V\,(C_{\text{f}} - C_{\text{i}})$. The **work done by the battery** then is $\Delta W_{\text{bat}} = V\Delta q = V^2\,(C_{\text{f}} - C_{\text{i}})$, in agreement with the result above.

15 10/28/2003 (Midterm exam)

16 10/30/2003

Continuity equation. Due to conservation of electric charge, it is generally

$$\nabla \cdot \mathbf{j}(\mathbf{x}, t) = -\frac{\partial}{\partial t}\rho(\mathbf{x}, t)$$

Magnetostatics. The range of validity of this theory covers static current distributions (*i.e.* the case $\nabla \cdot \mathbf{j} = 0$). Magnetic fields of current distributions that vary on time scales much slower than R/c are approximately described ($R = \max(|\mathbf{x} - \mathbf{x}'|)$ with the maximum taken over all source coordinates $\mathbf{x}'$ and observation coordinates of interest $\mathbf{x}'$).

Basics. Biot-Savart's law:

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\,\frac{d\mathbf{j} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \quad \text{and} \quad \mathbf{B} = \int d\mathbf{B} \quad .$$

To integrate, substitute $d\mathbf{j} = \mathbf{j}d^3x'$, $= \mathbf{K}da'$, $= Id\mathbf{l}'$, or $= q\mathbf{v_0}(\mathbf{t})\delta(\mathbf{x} - \mathbf{x}_0(t))d^3x'$ for volume currents, surface currents, line currents and moving charges, respectively. In the last case, which is only valid as long as $v_0 \ll c$, $\mathbf{x_0}(\mathbf{t})$ and $\mathbf{v}_0(t)$ characterize the particle trajectory.

Some examples amenable to direct calculation were mentioned (B-fields of line currents and circular-loop currents along loop axis; review if necessary).

Forces $\mathbf{F}$ and torques $\mathbf{N}$ of an external field $\mathbf{B}(\mathbf{x})$ on a current distribution:

$$d\mathbf{F} = d\mathbf{j} \times \mathbf{B} \quad \text{and} \quad \mathbf{F} = \int d\mathbf{F} \quad .$$

$$d\mathbf{N} = \mathbf{x} \times d\mathbf{F} \quad \text{and} \quad \mathbf{N} = \int d\mathbf{N} \quad .$$

To integrate, substitute $d\mathbf{j} = \mathbf{j}d^3x'$, $= \mathbf{K}da'$, $= Id\mathbf{l}'$, or $= q\mathbf{v_0}(\mathbf{t})\delta(\mathbf{x} - \mathbf{x}_0(t))d^3x'$ for volume currents, surface currents, line currents and moving charges, respectively. The last case is not restricted to $v_0 \ll c$.

Some examples amenable to direct calculation were mentioned (forces between two parallel line currents and between to general loop currents).

Differential equations for $\mathbf{B}$. Gauss's law and Ampere's law were derived from Biot-Savart's law and the right-hand-rule pertinent to the use of Ampere's law in integral form were discussed.

$$\begin{array}{lcl} \nabla \cdot \mathbf{B} = 0 & \Leftrightarrow & \oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} & \Leftrightarrow & \oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{j} \cdot d\mathbf{a} = \mu_0 I_{\text{enclosed}} \end{array} \tag{56}$$

Boundary conditions. Consider two regions 1 and 2 connected via a surface carrying a surface current $\mathbf{K}$. The respective B-fields at the interface are $\mathbf{B}_1$ and $\mathbf{B}_2$. Then, the boundary conditions

$$\begin{array}{rcl} \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) & = & 0 \\ \hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) & = & \mu_0 \mathbf{K} \quad , \end{array} \tag{57}$$

where $\mathbf{n}$ denotes the unit vector pointing from region 1 to region 2. Note the sensitivity of the second equation to the direction of $\hat{\mathbf{n}}$.
Exercise: Derive these conditions from Eqs. 56.

Vector potential. From Gauss's law it follows that there exists a vector field $\mathbf{A}(\mathbf{x})$ such that $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x})$.

There actually is an infinite number of valid vector potentials delivering the same magnetic field. Vector potentials $\mathbf{A}$ and $\mathbf{A}'$ for the same B-field are connected via a **gauge transformation**

$$\mathbf{A}' = \mathbf{A} + \nabla\psi \quad ,$$

with $\psi(\mathbf{x})$ being a well-behaved function. It was shown that there always exists a gauge transformation such that $\mathbf{A}'$ is in the **Coulomb gauge**:

$$\nabla \cdot \mathbf{A}' = 0$$

There also is an infinite number of vector potentials in the Coulomb gauge, since an $\mathbf{A}''$ with $\mathbf{A}'' = \mathbf{A}' + \nabla\psi$, $\mathbf{A}'$ in the Coulomb gauge, and ψ satisfying the Laplace equation, will also be in the Coulomb gauge.

The B-field, which - at least in classical physics - is the physically relevant observable, is independent of the gauge. Thus, there is no fundamental advantage of the Coulomb gauge. There is, however, a great technical advantage which is that in the Coulomb gauge a relatively simple differential equation for $\mathbf{A}$ applies. From Ampere's law it quickly follows that

$$\begin{array}{lll} \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} & & \Leftrightarrow \\ \Delta A_i = -\mu_0 j_i & \text{for} & i = 1, 2, 3 \end{array} \tag{58}$$

apply for vector potentials in the Coulomb gauge. In analogy with the solution of the Poisson equation in free space, the standard solution is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{59}$$

The solution is a vector potential in the Coulomb gauge. While this is by no means the only possible vector potential, it is sufficient to subsequently calculate the B-field (which is unique).

As an example, the vector potential of a circular current loop was calculated in detail.
Read Chapter 5.5 in textbook.

Multipole expansion of the B-fields of localized current distributions. Expanding $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in Eq. 59 using familiar methods, the leading term of the expansion is found to be the magnetic-dipole term; there is no monopole term because of $\nabla \cdot \mathbf{j} = 0$. The magnetic-dipole field with the (cartesian) magnetic-dipole moment $\mathbf{m}$ is given by

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3} \\ \mathbf{B}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \mathbf{m}}{|\mathbf{x}|^3} \\ \text{where} \quad \mathbf{m} &= \frac{1}{2} \int \mathbf{x}' \times \mathbf{j}(\mathbf{x}') d^3x' \end{aligned} \tag{60}$$

with $\hat{\mathbf{n}} = \mathbf{x}/|\mathbf{x}|$.

Examples. For loop currents with current I, the magnetic-dipole moment is $\mathbf{m} = \frac{I}{2} \oint \mathbf{x} \times d\mathbf{l}$, and for planar loop currents it is $\mathbf{m} = IA\hat{\mathbf{n}}$. There, A is the loop area and $\hat{\mathbf{n}}$ the area normal vector. The direction of $\hat{\mathbf{n}}$ is defined by the current direction and a right-hand rule.

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Multipole expansion of the vector potential: The objective of the method is to obtain an expansion of $\mathbf{A}(\mathbf{x})$ for localized current distributions observed at distances larger than the current distribution. **Methods**: To obtain spherical magnetic multipole moments, insert Eq. 29 into Eq. 59. The leading term is the dipole term ($l = 1$); there is no monopole term. Cartesian multipole moments are obtained by starting with a Taylor expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in Eq. 59 around $\mathbf{x}' = 0$. Properties of the dipole term were briefly reviewed.

Atomic and nuclear dipole moments: The connection between angular momentum and magnetic moment was discussed.

Miscellaneous useful relations: For spherical volumes of radius R containing all currents of interest it is

$$\int_{r<R} \mathbf{B}(\mathbf{x}) d^3x = \frac{2\mu_0}{3} \mathbf{m} \quad ,$$

where $\mathbf{m}$ is the dipole moment of the current distribution. The finding leads to a δ-function correction to the magnetic field of a point dipole located at the origin,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \left[\frac{3\hat{\mathbf{n}}(\mathbf{m} \cdot \hat{\mathbf{n}}) - \mathbf{m}}{|\mathbf{x}|^3} + \frac{8\pi}{3} \delta(\mathbf{x}) \right] \tag{61}$$

with $\hat{\mathbf{n}} = \mathbf{x}/|\mathbf{x}|$.

For spherical volumes of radius R with all currents located outside it is

$$\int_{r<R} \mathbf{B}(\mathbf{x}) d^3x = \frac{4\pi}{3} R^3 \mathbf{B}(0)$$

Interaction of localized current distributions with external fields that vary slowly within the volume of the current distributions: Expanding the **B**-field around a point inside the distribution, one finds expressions for the potential energy, the force and the torque on the leading multipole moment of the current distribution, which is the dipole moment **m** of Eq. 60. The equations are analogous to those in electrostatics (replace **E**, **p** with **B**, **m**).

As an example, the **magnetic-dipole hyperfine interaction** was discussed. It was pointed out that the δ-function term of Eq. 61 in the magnetic field of the electron magnetic moment is important to explain the S-state magnetic hyperfine structure.

18 11/6/2003

Magnetostatics in magnetically active media. Definitions of the macroscopic magnetization, the macroscopic **B**-field and the **H**-field,

$$\begin{aligned} \mathbf{M} &= \sum_i N_i \langle \mathbf{m}_i \rangle_{\text{time},100a_0} \\ \mathbf{B} &= \langle \mathbf{B}_{\text{micro}} \rangle_{\text{time},100a_0} \\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \quad , \end{aligned}$$

the fundamental equations

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{j}_{\text{free}} = \mathbf{j} \quad , \end{aligned} \tag{62}$$

the bound current density

$$\mathbf{j}_{\text{bound}} = \nabla \times \mathbf{M} \quad ,$$

and the applicable boundary conditions,

$$\begin{aligned} \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0 \\ \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K}_{\text{free}} = \mathbf{K} \quad , \end{aligned} \tag{63}$$

were reviewed. In the last two equations, two regions 1 and 2 connected via a surface carrying a free surface current **K** are considered. The respective fields at the interface are $\mathbf{B}_1$ and $\mathbf{B}_2$ and $\mathbf{H}_1$ and $\mathbf{H}_2$,

and **n** denotes the unit vector pointing from region 1 to region 2. Note that currents occurring in the macroscopic equations are free currents; the subscripts "free" are usually dropped. Free currents must not be confused with the currents in the microscopic equations (for **B**); the latter are free plus bound currents.

The significance of the fields (the fundamental **B**-field and the auxiliary **H**-field) and different types of linear magnetic behavior (paramagnetic, diamagnetic) and non-linear behavior (hysteresis, soft ferromagnetic, hard ferromagnetic) were discussed.

The behavior of **B**- and **H**-lines at a (free-)current-free interface between different linear, magnetically active media was sketched.

Methods to solve the magnetostatic equations.

- Linear material with known free current: Solve

  $$\nabla^2 \mathbf{A} = -\mu \mathbf{j} \quad ,$$

  and obtain $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{H} = \mathbf{B}/\mu$. An equation analogous to Eq. 59 and expansions of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ may be useful. If there is different domains with different μ, connect the solutions in the different domains using the boundary conditions Eq. 63 at the interfaces. From the fields, the magnetization is $\mathbf{M} = (\frac{\mu}{\mu_0} - 1)\mathbf{H}$, the (bound) volume magnetization current is $\mathbf{j}_M = \nabla \times \mathbf{M}$, and the surface magnetization current $\mathbf{K}_M = \mathbf{M} \times \hat{\mathbf{n}}$.

- Linear material with zero free current: Find the magnetostatic potential from

  $$\Delta \Phi_m = 0 \quad ,$$

  and obtain $\mathbf{H} = -\nabla \Phi_m$ and $\mathbf{B} = \mu \mathbf{H}$. Variable separation methods may be useful. If there is different domains with different μ, connect the solutions in the different domains using the boundary conditions Eq. 63 at the interfaces. From the fields, the magnetization is $\mathbf{M} = (\frac{\mu}{\mu_0} - 1)\mathbf{H}$, the (bound) volume magnetization current is $\mathbf{j}_M = \nabla \times \mathbf{M}$, and the surface magnetization current $\mathbf{K}_M = \mathbf{M} \times \hat{\mathbf{n}}$.

- Fixed magnetization and zero free current:
  Find the magnetostatic potential from

  $$\Delta \Phi_m = -\rho_M \quad ,$$

  where the volume "magnetic charge" is $\rho_M = -\nabla \cdot \mathbf{M}$. Under absence of boundary conditions it is

  $$\Phi_m = \frac{1}{4\pi} \int_{\text{all space}} \frac{\rho_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \quad .$$

  As always, variable separation methods and expansions of Green's functions may be useful. Obtain $\mathbf{H} = -\nabla \Phi_m$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. If there is different domains with different μ, connect the solutions in the different domains using the boundary conditions Eq. 63 at the interfaces.

  Special case: For a magnetized object of volume V and discontinuous surface ∂V, one can use the volume "charge" density $\rho_M = -\nabla \cdot \mathbf{M}$ and the surface "charge" density $\sigma_M = \mathbf{M} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ being the normal vector on the object, to write

$$\Phi_m = \frac{1}{4\pi}\int_{V\setminus\partial V}\frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{4\pi}\int_{\partial V}\frac{\sigma_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da' \quad .$$

Alternately, one may calculate the vector potential by inserting the (bound) volume magnetization current $\mathbf{j}_M = \nabla\times\mathbf{M}$ and the surface magnetization current $\mathbf{K}_M = \mathbf{M}\times\hat{\mathbf{n}}$ into Eq. 59, yielding

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V\setminus\partial V}\frac{\mathbf{j}_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{\mu_0}{4\pi}\int_{\partial V}\frac{\mathbf{K}_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da' \quad .$$

Reading: Sphere with homogeneous magnetization (various examples; Chapter 5.10 and 5.11). Magnetic shielding with permeable shell (Chapter 5.12).

Some solution methods for highly permeable materials.

- Current-free volume outside a highly permeable medium. A highly permeable medium with surface ∂V located in an external magnetic field acts as an equipotential surface for the magnetic potential outside the medium. A solution for the magnetic potential can be obtained by solving $\Delta\Phi_m = 0$ with the Dirichlet boundary condition $\Phi_m = 0$ on ∂V.
- Volume inside a linear highly permeable medium. Consider the two-dimensional case that the medium has a surface ∂V in the xy-plane, is invariant under z-translation, and carries a current $\mathbf{j}(x,y) = j_z(x,y)\hat{\mathbf{z}}$. Then, the vector potential can be chosen as $\mathbf{A}(\mathbf{x}) = A_z(x,y)\hat{\mathbf{z}}$. The problem reduces to a Poisson equation $\Delta A_z(x,y) = -\mu j_z(x,y)$ with Dirichlet boundary condition $A_z(x,y) = 0$ on ∂V. Solution methods known from electrostatics can be employed, including numerical ones such as the iteration method Eq 1.82 of the textbook (set $g(x,y) = \mu j_z(x,y)$).

19 11/11/2003

Faraday's law.

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} \quad \Leftrightarrow \quad \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Assuming Galilean invariance of the magnetic field, a transformation law for the electric field was derived from Faraday's law in integral form. In a frame moving with velocity $\mathbf{v_0}$ with respect to an inertial frame in which the electric field is $\mathbf{E}$ and the magnetic field is $\mathbf{B}$, the electric field is $\mathbf{E}' = \mathbf{E} + \mathbf{v_0}\times\mathbf{B}$. This is correct in first order in v_0/c (compare with Eq. 11.149 in Jackson). In the context of this proof, learn about the **convective derivative**, explained in class and in the textbook.

It was derived that **the magnetostatic energy**, including magnetization energy, is

$$W = \int_V d^3x \int_{\mathbf{B}=0}^{\mathbf{B}(\mathbf{x})} \mathbf{H}(\mathbf{B},\,\text{path})\cdot\delta\mathbf{B}$$

This equation is generally valid (including nonlinear materials and materials with hysteresis).

In **linear materials**, it is

$$W = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}d^3x = \frac{1}{2}\int_V \mathbf{j}\cdot\mathbf{A}d^3x \quad .$$

The energy of a **linear** object brought from a field-free region into a region with magnetic fields produced by **fixed currents** and initial values of the field $\mathbf{B}_{\text{initial}}(\mathbf{x})$ and final values of the magnetization $\mathbf{M}_{\text{final}}(\mathbf{x})$ is

$$W = \frac{1}{2}\int_V \mathbf{B}_{\text{initial}}(\mathbf{x})\cdot\mathbf{M}_{\text{final}}(\mathbf{x})d^3x \quad .$$

The component of the force on such an object corresponding to a generalized coordinate ξ is

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_{\text{currents fixed}} \quad .$$

These equations have strong analogies with equations from electrostatics, which may be exploited for memorization.

For systems of **current loops with currents** I_i, the magnetostatic energy

$$W = \frac{1}{2}\int_V \mathbf{j}\cdot\mathbf{A}d^3x = \frac{\mu_0}{8\pi}\int_V\int_V \frac{\mathbf{j}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x\,d^3x'$$

can be rewritten by substituting $\mathbf{j}(\mathbf{x})d^3x \to I_i d\mathbf{l}_i$ and $\mathbf{j}(\mathbf{x}')d^3x' \to I_j d\mathbf{l}_j$ into a practical form:

$$\begin{aligned} W &= \frac{1}{2}\sum_i L_i I_i^2 + \frac{1}{2}\sum_{i,j(i\neq j)} M_{ij} I_i I_j = \frac{1}{2}\sum_i L_i I_i^2 + \sum_{i>j} M_{ij} I_i I_j \\ L_i &= \frac{\mu_0}{4\pi}\oint_i\oint_i \frac{d\mathbf{l}_i\cdot d\mathbf{l}_i'}{|\mathbf{x}_i-\mathbf{x}_i'|} \\ M_{ij} &= \frac{\mu_0}{4\pi}\oint_i\oint_j \frac{d\mathbf{l}_i\cdot d\mathbf{l}_j}{|\mathbf{x}_i-\mathbf{x}_j|} \end{aligned}$$

There, L_i are self-inductances and M_{ij} mutual inductances. Obviously, the self energy of loop i is $\frac{1}{2}L_i I_i^2$. The M_{ij} are mutual inductances that describe the loop interaction energies, which are $M_{ij} I_i I_j$.

The voltage induced at loop i due to current variations in any loop, including its own current, are given by

$$U_i = -\sum_j M_{ij}\frac{d}{dt}I_j \quad .$$

All equations can be written in a comprehensive matrix form, as shown in class.

Magnetic-field diffusion. As well known, the spread of magnetic-field lines in conducting media is hampered by induction and Lenz's rule. In effect, in many situations magnetic-field lines spread much slower than the speed of light, and the time variation of the fields is slow enough that the displacement current in Ampere's law pays no significant role. From Maxwell's equations (less the displacement current) and Ohm's law, $\mathbf{j} = \sigma\mathbf{E}$ with conductivity σ, it then follows for media with uniform and time-independent permeability μ

$$\nabla^2\mathbf{A}(\mathbf{x},t) = \mu\sigma\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t)$$

This is a diffusion equation. The same equation applies to the electric field $\mathbf{E}(\mathbf{x},t)$. It was then estimated that the typical penetration time of B-fields through a system of spatial size L is

$$\tau \sim \mu\sigma L^2 \quad .$$

If the conductivity is homogeneous, too, the same diffusion equation applies to $\mathbf{B}(\mathbf{x},t)$ and $\mathbf{j}(\mathbf{x},t)$.

Example. The case of a magnetic field with harmonic time dependence incident on a conducting medium was discussed (Jackson, p220f). The fields and the resultant Ohm heating power in the medium were given, and the significance of the skin depth,

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \quad ,$$

was explained. **Study this example and derive the results**.

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Maxwell's equations. It was shown how to properly add the displacement current $\frac{\partial}{\partial t}\mathbf{D}$ into Ampere's law. The resultant complete set of Maxwell's equations in media was noted, and their temporal, spatial and energetic ranges of validity were discussed (the microscopic equations are valid over wider ranges).

Partial solution via potentials. The homogeneous equations are automatically solved by writing

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) &= \nabla\times\mathbf{A}(\mathbf{x},t)\\ \mathbf{E}(\mathbf{x},t) &= -\nabla\Phi(\mathbf{x},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t)\end{aligned}$$

The scalar potential Φ and the vector potential $\mathbf{A}$ follow a smaller set of coupled partial differential equations, which in **free space** ($\mathbf{D} = \epsilon_0\mathbf{E}$, $\mathbf{H} = \mu_0^{-1}\mathbf{B}$) reads

$$\begin{aligned} \nabla^2\Phi(\mathbf{x},t) + \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}(\mathbf{x},t)\right) &= -\frac{\rho(\mathbf{x},t)}{\epsilon_0} \\ \nabla^2\mathbf{A}(\mathbf{x},t) - \frac{\partial^2}{c^2\partial t^2}\mathbf{A}(\mathbf{x},t) - \nabla\left(\nabla\cdot\mathbf{A}(\mathbf{x},t) + \frac{\partial}{c^2\partial t}\Phi(\mathbf{x},t)\right) &= -\mu_0\mathbf{j}(\mathbf{x},t) \end{aligned} \tag{64}$$

where the vacuum velocity of light $c = 1/\sqrt{\epsilon_0\mu_0}$. Note that Eq. 64 applies to potentials in any gauge.

Gauge transformations and Lorentz gauge. Using a gauge transformation with an arbitrary scalar function ψ, one can introduce new sets of potentials that yield the same fields as the original ones and are therefore equally valid:

$$\begin{aligned} \mathbf{A}'(\mathbf{x},t) &= \mathbf{A}(\mathbf{x},t) + \nabla\psi(\mathbf{x},t) \\ \Phi'(\mathbf{x},t) &= \Phi(\mathbf{x},t) - \frac{\partial}{\partial t}\psi(\mathbf{x},t) \end{aligned}$$

For any set of potentials $\Psi, \mathbf{A}$, the scalar function ψ can be chosen such that it satisfies an inhomogeneous wave equation

$$\left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\psi(\mathbf{x},t) = -\left[\nabla\cdot\mathbf{A}(\mathbf{x},t) + \frac{\partial}{c^2\partial t}\Phi(\mathbf{x},t)\right] \quad ,$$

which is solvable (see Eq. 6.32ff), and yields gauge-transformed potentials with

$$\nabla\cdot\mathbf{A}'(\mathbf{x},t) + \frac{\partial}{c^2\partial t}\Phi'(\mathbf{x},t) = 0$$

Potentials satisfying this condition are said to be in the **Lorenz gauge**. It is thus always possible to adopt the Lorenz gauge.

Note. Gauge transformations of potentials that are already in the Lorenz gauge with functions ψ that solve the *homogeneous* wave equation yield new potentials that are also in the Lorenz gauge.

In the Lorenz gauge, the equations for the potentials reduce to inhomogeneous wave equations,

$$\begin{aligned} \left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\Phi(\mathbf{x},t) &= -\frac{\rho(\mathbf{x},t)}{\epsilon_0} \\ \left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\mathbf{A}(\mathbf{x},t) &= -\mu_0\mathbf{j}(\mathbf{x},t) \qquad \text{in Lorenz gauge} \end{aligned} \tag{65}$$

The equations for Φ and $\mathbf{A}$ are automatically decoupled.

Note. One can always adopt the Lorenz gauge. The solution of Eq. 65 will then, of course, yield potentials in the Lorenz gauge. While these potentials are not unique in any way, the fields derived from them are.

Coulomb gauge. As shown before, one may also require $\nabla \cdot \mathbf{A} = 0$. Any set of potentials Ψ, $\mathbf{A}$ can be gauge-transformed into the Coulomb gauge by finding a ψ that satisfies the Poisson equation $\Delta\psi = -\nabla \cdot \mathbf{A}$.

In the Coulomb gauge, the scalar potential is identical with the instantaneous Coulomb potential (hence the name Coulomb gauge), and the vector potential follows from a wave equation the inhomogeneity of which contains a term depending on the scalar potential:

$$\begin{aligned} \nabla^2\Phi(\mathbf{x},t) &= -\frac{\rho(\mathbf{x},t)}{\epsilon_0} \quad \Rightarrow \quad \Psi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ \left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\mathbf{A}(\mathbf{x},t) &= -\mu_0\mathbf{j}(\mathbf{x},t) + \frac{1}{c^2}\nabla\frac{\partial}{\partial t}\Psi(\mathbf{x},t) \qquad \text{in Coulomb gauge} \end{aligned} \tag{66}$$

Thus, if one intends to use these equations to find Ψ and $\mathbf{A}$, the scalar potential Ψ should be determined first.

The Coulomb gauge is particularly useful if there are only radiation fields (hence the alternate name radiation gauge). With $\rho = 0$ and $\mathbf{j} = 0$ it is

$$\begin{aligned} \Phi(\mathbf{x},t) &= 0 \\ \left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\mathbf{A}(\mathbf{x},t) &= 0 \qquad \text{Coulomb gauge without sources} \end{aligned} \tag{67}$$

Thus, only the vector potential needs to be determined, and it follows from a homogeneous wave equation.

Coulomb gauge using transverse current. The equations for Ψ and $\mathbf{A}$ in the Coulomb gauge can be decoupled using a decomposition of the current $\mathbf{j}(\mathbf{x},t)$ into a longitudinal part $\mathbf{j}_l(\mathbf{x},t)$ and a transverse part $\mathbf{j}_t(\mathbf{x},t)$ with $\nabla \times \mathbf{j}_l(\mathbf{x},t) = 0$ and $\nabla \cdot \mathbf{j}_t(\mathbf{x},t) = 0$ (hence the alternate name transverse gauge for the Coulomb gauge). According to the Helmholtz theorem, such a decomposition always exists. Explicitly, one may obtain the transverse current $\mathbf{j}_t$ and the longitudinal current $\mathbf{j}_l$ from

$$\begin{aligned} \mathbf{j}(\mathbf{x},t) &= \mathbf{j}_l(\mathbf{x},t) + \mathbf{j}_t(\mathbf{x},t) \\ \mathbf{j}_t(\mathbf{x},t) &= \frac{1}{4\pi}\nabla \times \nabla \times \int_V \frac{\mathbf{j}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ \mathbf{j}_l(\mathbf{x},t) &= -\frac{1}{4\pi}\nabla \int_V \frac{\nabla' \cdot \mathbf{j}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \end{aligned}$$

It then becomes

$$
\begin{aligned}
\nabla^2\Phi(\mathbf{x},t) &= -\frac{\rho(\mathbf{x},t)}{\epsilon_0} \\
\left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\mathbf{A}(\mathbf{x},t) &= -\mu_0\mathbf{j}_t(\mathbf{x},t) \\
\text{where}\quad \mathbf{j}_t(\mathbf{x},t) &= \frac{1}{4\pi}\nabla\times\nabla\times\int_V \frac{\mathbf{j}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\
&\quad \text{Coulomb gauge using transverse current}
\end{aligned}
\tag{68}
$$

Noting that the solution of Maxwell's equations (both directly or via potentials) often requires the solution of inhomogeneous wave equations, the Green's function of the wave equation will be a useful instrument. It was derived in class for systems without boundary conditions and constant, frequency-independent ϵ and μ (in most cases of interest, $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

Helmholtz equation and its Green's function. Inserting the temporal Fourier transform of the solution $\psi(\mathbf{x},t)$ and the inhomogeneity $f(\mathbf{x},t)$ into the wave equation, we find, using the orthogonality condition $\int_{-\infty}^{\infty}\exp(\mathrm{i}(\omega-\omega')t)dt = 2\pi\delta(\omega-\omega')$,

$$
\begin{aligned}
\left[\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right]\psi(\mathbf{x},t) &= -4\pi f(\mathbf{x},t) \\
\psi(\mathbf{x},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(\mathbf{x},\omega)\exp(-\mathrm{i}\omega t)d\omega \quad \text{with} \quad \tilde{\psi}(\mathbf{x},\omega) = \int_{-\infty}^{\infty}\psi(\mathbf{x},t)\exp(\mathrm{i}\omega t)dt \\
f(\mathbf{x},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\mathbf{x},\omega)\exp(-\mathrm{i}\omega t)d\omega \quad \text{with} \quad \tilde{f}(\mathbf{x},\omega) = \int_{-\infty}^{\infty}f(\mathbf{x},t)\exp(\mathrm{i}\omega t)dt \\
\Rightarrow \quad \left[\nabla^2 + k^2\right]\tilde{\psi}(\mathbf{x},\omega) &= -4\pi\tilde{f}(\mathbf{x},\omega) \quad .
\end{aligned}
\tag{69}
$$

The last equation is an inhomogeneous Helmholtz equation; $k = \frac{\omega}{c}$. Its free-space Green's function, defined as

$$
\left[\nabla^2 + k^2\right]G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta^3(\mathbf{x}-\mathbf{x}') \quad , \tag{70}
$$

depends only on k and $R := |\mathbf{x}-\mathbf{x}'|$ and is

$$
G_k(R) = A\frac{\exp(\mathrm{i}kR)}{R} + (1-A)\frac{-\exp(\mathrm{i}kR)}{R} =: AG^+(R) + (1-A)G^-(R)
$$

with constant A. This result has been derived in class, and it has been explained why $G^+(R)$ is associated with an outgoing wave (from a source location $\mathbf{x}'$), and $G^-(R)$ with an ingoing wave.

Note: Using the Green's function, the solution to Eq. 69 in free space is

$$
\tilde{\psi}(\mathbf{x},\omega) = \int_V \tilde{f}(\mathbf{x}',\omega)G_k(\mathbf{x},\mathbf{x}')d^3x' \quad ,
$$

because

$$
\begin{aligned}
\left[\nabla^2 + k^2\right] \tilde{\psi}(\mathbf{x},\omega) &= \left[\nabla^2 + k^2\right] \int_{-\infty}^{\infty} \tilde{f}(\mathbf{x}',\omega) G_k(\mathbf{x},\mathbf{x}') d^3x' \\
&= \int_{-\infty}^{\infty} \tilde{f}(\mathbf{x}',\omega) \left\{ \left[\nabla^2 + k^2\right] G_k(\mathbf{x},\mathbf{x}') \right\} d^3x' \\
&= \int_{-\infty}^{\infty} \tilde{f}(\mathbf{x}',\omega) (-4\pi) \delta^3(\mathbf{x}-\mathbf{x}')\, d\omega \\
&= -4\pi \tilde{f}(\mathbf{x},\omega)
\end{aligned}
$$

Green's function of wave equation. Per def., the Green's function satisfies

$$
\left[\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right] G(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t')
$$

Inserting temporal Fourier transforms wrt. to t for $G(\mathbf{x},t,\mathbf{x}',t')$ and $\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t')$ and using the orthogonality of the $\exp(\mathrm{i}\omega t)$, the reduced Green's function for $k = \omega/c$ is seen to satisfy

$$
\begin{aligned}
\left[\nabla^2 + k^2\right] G_k(\mathbf{x},\mathbf{x}') &= -4\pi\delta^3(\mathbf{x}-\mathbf{x}') \exp(\mathrm{i}\omega t') \\
\left[\nabla^2 + k^2\right] \left(\frac{G_k(\mathbf{x},\mathbf{x}')}{\exp(\mathrm{i}\omega t')} \right) &= -4\pi\delta^3(\mathbf{x}-\mathbf{x}') \quad . \qquad (71)
\end{aligned}
$$

Comparison with Eq. 70ff shows that

$$
G_k(\mathbf{x},\mathbf{x}') = G_k^{\pm}(R) = \frac{\exp(\pm \mathrm{i}kR)}{R} \exp(\mathrm{i}\omega t') \quad .
$$

The inverse Fourier transform then yields $G^{\pm}(R, t-t')$:

$$
\begin{aligned}
G^{\pm}(R,t-t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G_k^{\pm}(R) \exp(-\mathrm{i}\omega t) d\omega \\
&= \frac{1}{2\pi} \frac{1}{R} \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\omega (t' - t \pm \frac{R}{c}) \right] d\omega \\
&= \frac{\delta(t' - \left[t \mp \frac{R}{c}\right])}{R} \quad . \qquad (72)
\end{aligned}
$$

.

There, the upper signs correspond to the **retarded Green's function**, in which a source at location $\mathbf{x}'$ and time t' produces an outgoing wave that arrives at the observation point $\mathbf{x}$ at a delayed time $t = t' + R/c$. The lower signs correspond to the **advanced Green's function**, in which a signal observed at an observation point $\mathbf{x}$ at a time t produces an ingoing wave that converges towards a source at location $\mathbf{x}'$ and time $t' = t + R/c$. Physical interpretations of the two types of Green's functions, consistent with causality, will be given later.

With the Green's functions, we can write down two particular solutions of

$$\left[\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right] \psi(\mathbf{x}, t) = -4\pi f(\mathbf{x}, t) \quad ,$$

namely

$$\begin{aligned}
\psi^{\pm}(\mathbf{x}, t) &= \int_V \int_{t'=-\infty}^{\infty} f(\mathbf{x}', t') G^{\pm}(\mathbf{x}, t, \mathbf{x}', t') d^3x' dt' \\
\psi^{\pm}(\mathbf{x}, t) &= \int_V \int_{t'=-\infty}^{\infty} f(\mathbf{x}', t') \frac{\delta(t' - \left[t \mp \frac{R}{c}\right])}{R} d^3x' dt' \\
\psi^{\pm}(\mathbf{x}, t) &= \int_V \frac{f(\mathbf{x}', t' = \left[t \mp \frac{R}{c}\right])}{R} d^3x' \quad . \qquad (73)
\end{aligned}$$

There, in the last line the time integral over the source time t' have been evaluated using the temporal δ-function part in the Green's functions.

The following notations are used. For solutions involving retarded particular solutions constructed with the retarded Green's function we write

$$\psi(\mathbf{x}, t) = \psi_0(\mathbf{x}, t) + \psi^{+}(\mathbf{x}, t) = \psi_0(\mathbf{x}, t) + \int_V \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{R} d^3x' \quad . \qquad (74)$$

The notation implies that the time variable t' is not an integration variable. It is to be constructed from the observation coordinates $\mathbf{x}$ and t and from the source location $\mathbf{x}'$, over which we take the integral, following

$$t' = t - \frac{R}{c} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} < t \quad .$$

In Eq. 74 we have added a solution $\psi_0(\mathbf{x}, t)$ of the homogeneous wave equation to match any boundary conditions, if needed (discussed later).

Similarly, for solutions involving particular solutions constructed with the advanced Green's function we write

$$\psi(\mathbf{x}, t) = \psi_0(\mathbf{x}, t) + \psi^{-}(\mathbf{x}, t) = \psi_0(\mathbf{x}, t) + \int_V \frac{[f(\mathbf{x}', t')]_{\text{adv}}}{R} d^3x' \quad . \qquad (75)$$

Here, t' is to be constructed from the observation coordinates $\mathbf{x}$ and t and from the source location $\mathbf{x}'$ using

$$t' = t + \frac{R}{c} = t + \frac{|\mathbf{x} - \mathbf{x}'|}{c} > t \quad .$$

Again, in Eq. 75 we have added a solution $\psi_0(\mathbf{x}, t)$ of the homogeneous wave equation to match any boundary conditions (discussed later).

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The physical significance of retarded and advanced Green's functions was discussed. Generally,

$$\psi^{\pm}(\mathbf{x},t) = \psi_0(\mathbf{x},t) + \int_V G^{\pm}(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',\mathbf{t}') d^3x' dt' \quad . \tag{76}$$

- If $\psi_0(\mathbf{x},t)$ in Eq. 76 represents a solution of the homogeneous wave equation **before** sources turn on, the **retarded** Green's function G^+ must be used. In that case, during and at any time after the activity of the sources the integral term in Eq. 76 generates the source-induced waves. Due to the retardation condition $t' = t - |\mathbf{x} - \mathbf{x}'|/c$, the relation between observation coordinates $(ct, \mathbf{x})$ and source coordinates $(ct', \mathbf{x}')$ is explicitly causal, meaning that in vacuum signals propagate at the velocity of light.
- If $\psi_0(\mathbf{x},t)$ in Eq. 76 represents a solution of the homogeneous wave equation **after** the sources have turned off, $\psi_0(\mathbf{x},t)$ already includes the source-induced waves. In that case, the **advanced** Green's function must be used. The integral term in Eq. 76 then annihilates the source-induced waves at observation times before the sources turn on. Thus, the effect of the advanced Green's function is to keep source-induced waves from propagating into the past of the source time.

In any case, both types of Green's functions - retarded and advanced - are, when used properly, consistent with the principle of causality.

It was outlined how one can find from either the Maxwell equations or form the potentials Φ, $\mathbf{A}$ the following wave equations (in vacuum):

$$\left[\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right] \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\epsilon_0}\left[-\nabla\rho - \frac{\partial}{c^2\partial t}\mathbf{j}\right] \\ -\mu_0 \left[\nabla \times \mathbf{j}\right] \end{pmatrix} \tag{77}$$

Using the previously found expressions for the Green's functions for the case that there are no fields prior to source activation, it is

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[-\nabla'\rho(\mathbf{x}',t') - \frac{\partial}{c^2 \partial t'}\mathbf{j}(\mathbf{x}',t')\right]_{\text{ret}} \\ \mathbf{B}(\mathbf{x},t) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[\nabla' \times \mathbf{j}(\mathbf{x}',t')\right]_{\text{ret}} \end{aligned} \tag{78}$$

Driven by a desire to remove the ∇ from these equations, it was first shown that for functions $f(\mathbf{x}',t')$ and vector fields $\mathbf{f}(\mathbf{x}',t')$

$$\begin{aligned} \left[\nabla' f(\mathbf{x}',t')\right]_{\text{ret}} &= \nabla' \left[f(\mathbf{x}',t')\right]_{\text{ret}} - \frac{\hat{\mathbf{R}}}{c}\left[\frac{\partial}{\partial t'} f(\mathbf{x}',t')\right]_{\text{ret}} \\ \left[\nabla' \times \mathbf{f}(\mathbf{x}',t')\right]_{\text{ret}} &= \nabla' \times \left[\mathbf{f}(\mathbf{x}',t')\right]_{\text{ret}} - \frac{\hat{\mathbf{R}}}{c} \times \left[\frac{\partial}{\partial t'} \mathbf{f}(\mathbf{x}',t')\right]_{\text{ret}} \quad , \end{aligned}$$

from which it follows that

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{\hat{\mathbf{R}}}{R^2}\left[\rho(\mathbf{x}',t')\right]_{\rm ret} + \frac{\hat{\mathbf{R}}}{cR}\left[\frac{\partial}{\partial t'}\,\rho(\mathbf{x}',t')\right]_{\rm ret} - \frac{1}{c^2R}\left[\frac{\partial}{\partial t'}\,\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\right\} \\
\mathbf{B}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int d^3x' \left\{ \left[\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\times\frac{\hat{\mathbf{R}}}{R^2} + \left[\frac{\partial}{\partial t'}\,\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\times\frac{\hat{\mathbf{R}}}{cR}\right\} \quad . \qquad (79)
\end{aligned}$$

Since $t'_{\rm ret} = t - R/c$ and $R = |\mathbf{x}-\mathbf{x}'|$ does not include time variables, this equals

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{\hat{\mathbf{R}}}{R^2}\left[\rho(\mathbf{x}',t')\right]_{\rm ret} + \frac{\hat{\mathbf{R}}}{cR}\frac{\partial}{\partial t}\,\left[\rho(\mathbf{x}',t')\right]_{\rm ret} - \frac{1}{c^2R}\frac{\partial}{\partial t}\,\left[\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\right\} \\
\mathbf{B}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int d^3x' \left\{ \left[\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\times\frac{\hat{\mathbf{R}}}{R^2} + \frac{\partial}{\partial t}\,\left[\mathbf{j}(\mathbf{x}',t')\right]_{\rm ret}\times\frac{\hat{\mathbf{R}}}{cR}\right\} \quad . \qquad (80)
\end{aligned}$$

These expressions were then specified to the field of a moving point charge q with trajectory $\mathbf{r}_0(t)$ and velocity $\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}_0(t)$. Then,

$$\begin{aligned}
\rho(\mathbf{x}',t') &= q\,\delta^3(\mathbf{x}'-\mathbf{r}_0(t')) \\
\mathbf{j}(\mathbf{x}',t') &= q\,\mathbf{v}(t')\,\delta^3(\mathbf{x}'-\mathbf{r}_0(t'))
\end{aligned}$$

Some hints were given for homework problem 6.2, which is to obtain the fields

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\rm ret} + \frac{\partial}{c\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\rm ret} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\rm ret}\right\} \\
\mathbf{B}(\mathbf{x},t) &= \frac{\mu_0\,q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\rm ret} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\rm ret}\right\} \quad . \qquad (81)
\end{aligned}$$

There, $\kappa = \left[1 - \frac{\mathbf{v}\cdot\hat{\mathbf{R}}}{c}\right]_{\rm ret}$ results from a Jacobi determinant that incurs when the spatial integral in Eq. 80 is executed using the $\delta^3(\mathbf{x}'-\mathbf{r}_0(t'))$. It was noted that the independent variables of $\mathbf{R}$ in Eq. 81 are $\mathbf{x}$ and t.

Eq. 81 was discussed using space-time diagrams. The initial steps of a field calculation for a point charge moving with constant velocity along the z-axis were presented. To see the fields for that specific case, refer to Sec. 11.10.

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Conservation laws for energy. Poynting theorem's for linear materials with time-independent (= frequency-independent) and real ϵ and μ was derived and written in differential and integral forms,

$$\begin{aligned} \int_V \mathbf{E}\cdot\mathbf{j}d^3x + \int_V \frac{\partial}{\partial t}ud^3x &= -\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x =: -\int_V \nabla\cdot\mathbf{S}d^3x = -\int_{\partial V}\mathbf{S}\cdot d\mathbf{a} \\ \mathbf{E}\cdot\mathbf{j} + \frac{\partial}{\partial t}u &= -\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\nabla\cdot\mathbf{S} \end{aligned} \tag{82}$$

where $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ is the Poynting vector and u is the electromagnetic energy density, which includes all polarization energies and equals $u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$.

All terms in this energy conservation law have physical meanings, which were discussed. The power density of Ohm heating is $\mathbf{E}\cdot\mathbf{j}$, the rate at which the EM energy density varies is $\frac{\partial}{\partial t}u$, and the power density due to radiation is $\nabla\cdot S$. The significance of the latter largely is that the radiation intensity (= energy per area per time) through a surface element $d\mathbf{a}$ is $\mathbf{S}\cdot d\mathbf{a}$. In Eq. 82, $d\mathbf{a}$ points out of the volume of interest.

Note the restrictions of this result. In particular, real ϵ and μ implies that there are no polarization losses in the medium.

Poynting theorem's for linear materials with frequency-dependent $\epsilon(\omega)$ and $\mu(\omega)$. In the case of frequency-dependence (dispersion), $\epsilon(\omega)$ and $\mu(\omega)$ necessarily have imaginary parts, as dictated by the Kramers-Kronig relations discussed later. Assuming that the spectral intensity of the fields cover only a small ω-range centered around a carrier frequency ω_0, a fairly simple energy conservation law similar to Eq. 82 can be derived (Eqs. 6.126f in Jackson). A notable new feature is the occurrence of a new term accounting for energy losses caused by phase lags between fields and polarizations ("polarization losses"; second term on the rhs of Eq. 6.127).

Conservation laws for momentum. The derivation of a conservation law for the momentum of particles and fields in vacuum was sketched. The result,

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}})_\alpha = \oint_{\partial V}\sum_\beta T_{\alpha\beta}\, n_\beta\, da \tag{83}$$

was interpreted in detail. There, α and β are indices for cartesian coordinate directions, $\mathbf{P}_{\text{mech}} = \sum_i \mathbf{p}_i$ is the sum of all momenta $\mathbf{p}_i$ of the particles, labeled i, inside the volume of interest,

$$\mathbf{P}_{\text{field}} = \epsilon_0\int_V \mathbf{E}\times\mathbf{B}d^3x = \frac{1}{c^2}\int_V \mathbf{S}d^3x$$

is the momentum associated with the electromagnetic field,

$$\epsilon_0\mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

is the momentum density associated with the electromagnetic field, n_β are the cartesian components of the normal vector of the surface ∂V of the volume of interest V, and

$$T_{\alpha\beta} = \epsilon_0\left(E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(|\mathbf{E}|^2 + c^2\,|\mathbf{B}|^2\right)\delta_{\alpha\beta}\right)$$

are the elements of the Maxwell stress tensor. To develop some intuition of the stress tensor, the effect of the rhs of Eq. 83 was compared with hydrostatic pressure. Thereby, note that the hydrostatic pressure in tensor notation would be diagonal with all diagonal elements being equal.

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Energy conservation in the case of harmonic fields. Fields and sources are written as

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{x})\exp(-\mathrm{i}\omega t)\right] \quad ,$$

where the $\mathbf{E}$ on the left is real and dependent on $\mathbf{x}$ and t, while the $\mathbf{E}$ on the right is complex and dependent on $\mathbf{x}$ only. Similar arrangements are made for the other fields and the sources. The Maxwell equations then yield the following set of equations for the complex field quantities and sources:

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0 \\ \nabla\cdot\mathbf{D} &= \rho \\ \nabla\times\mathbf{H}+\mathrm{i}\omega\mathbf{D} &= \mathbf{j} \\ \nabla\times\mathbf{E}-\mathrm{i}\omega\mathbf{B} &= 0 \quad . \end{aligned} \tag{84}$$

Upon solution, real fields are obtained from

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{2}\left(\mathbf{E}(\mathbf{x})\exp(-\mathrm{i}\omega t) + c.c.\right) = \frac{1}{2}\left(\mathbf{E}(\mathbf{x})\exp(-\mathrm{i}\omega t) + \mathbf{E}^*(\mathbf{x})\exp(+\mathrm{i}\omega t)\right) \quad \text{etc.}$$

The energy conservation laws involve **products of field quantities**. Since **for harmonic fields** the cycle-averages of such products follow

$$\langle\mathbf{E}(\mathbf{x},t)\cdot\mathbf{j}(\mathbf{x},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{x})\cdot\mathbf{J}^*(\mathbf{x})\right] = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{x})^*\cdot\mathbf{J}(\mathbf{x})\right] \quad \text{etc.},$$

expressions involving this and and other (including cross) products of **real** fields can be translated into forms for the complex spatial parts of the fields/sources by complex-conjugating one of the two factors and multiplying with 1/2. It does not matter which one of the variables in products is complex-conjugated. Physically meaningful laws are extracted by taking the real part.

The following definitions are common:

- Complex Poynting vector: $\mathbf{S}(\mathbf{x}) = \frac{1}{2}\mathbf{E}(\mathbf{x})\times\mathbf{H}^*(\mathbf{x})$
- Complex magnetic energy density: $u_m(\mathbf{x}) = \frac{1}{4}\mathbf{B}(\mathbf{x})\cdot\mathbf{H}^*(\mathbf{x})$
- Complex electric energy density: $u_e(\mathbf{x}) = \frac{1}{4}\mathbf{E}(\mathbf{x})\cdot\mathbf{D}^*(\mathbf{x})$

Application to energy conservation. The Ohm heating power density, $\mathbf{E}(\mathbf{x},t)\cdot\mathbf{j}(\mathbf{x},t)$, has been seen to follow

$$\mathbf{E}\cdot\mathbf{j}+\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{E}\frac{\partial}{\partial t}\mathbf{D}+\mathbf{H}\frac{\partial}{\partial t}\mathbf{B}\quad . \tag{85}$$

Inserting complex fields, taking the time derivatives, and then applying the described translation procedure yields

$$\frac{1}{2}\mathbf{E}\cdot\mathbf{j}^*+\frac{1}{2}\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\frac{1}{2}\mathbf{E}\cdot[-\mathrm{i}\omega\mathbf{D}]^*+\frac{1}{2}[-\mathrm{i}\omega\mathbf{B}]\cdot\mathbf{H}^*=0\quad .$$

Using the above definitions for the complex Poynting vector and the complex field energy densities, we obtain the complex Poynting theorem,

$$\frac{1}{2}\mathbf{E}\cdot\mathbf{j}^*+\nabla\cdot\mathbf{S}+2\mathrm{i}\omega\ [u_e-u_m]=0 \tag{86}$$

A physical law is extracted by taking the real part. Assuming complex ϵ and μ and integrating over space, it is seen that

$$\int_V\frac{1}{2}\mathrm{Re}\left[\mathbf{E}\cdot\mathbf{j}^*\right]d^3x+\int_{\partial V}\mathbf{S}\cdot d\mathbf{a}+\int_V\frac{\omega}{2}\left[\mathrm{Im}(\epsilon)\left|\mathbf{E}\right|^2+\mathrm{Im}(\mu)\left|\mathbf{H}\right|^2\right]d^3x=0 \tag{87}$$

The physical meaning of the terms was discussed (Ohm heating, radiation transport, polarization loss; same order as in equation). The steady-state-nature of the situation described by Eq. 87 was stressed.

Exercise. Show that for the possible choices of which field variables in the products in Eq. 85 are complex-conjugated the extracted physical law always is Eq. 87.

Reading. Determination of the impedance of a linear circuit (Sec. 6.9).

Transformation properties of the laws of classical electromagnetism. The objective of this unit is to show the invariance of classical electromagnetism under rotation (not discussed explicitly but true), spatial inversion and time reversal.

The theory of orthogonal transformations was briefly reviewed. Detailed elaborations can be found in linear algebra textbooks.

Definitions. We assume that a system is actively transformed using a rotation described by a matrix $\mathcal{A}$ with elements $a_{\alpha\beta}$. The transformed position vectors are denoted $\mathbf{x}'=\mathcal{A}\mathbf{x}$; transformed functions of transformed position vectors are also identified with a '.

- Scalar = tensor of rank 0 = a scalar function with $S'(\mathbf{x}')=S(\mathbf{x})$
  Examples: electric potential, charge (density), divergence of vectors as defined in the next item.
- Vector = tensor of rank 1 = a vector field satisfying $\mathbf{V}'(\mathbf{x}')=\mathcal{A}\mathbf{V}(\mathbf{x})$
  Examples: position, momentum, gradient of scalars as defined in the first item.
- Tensor of rank 2 = a quantity the components of which transform as $T'_{\alpha\beta}(\mathbf{x}')=\sum_{\gamma\delta}a_{\alpha\gamma}a_{\beta\delta}T_{\alpha\beta}(\mathbf{x})$
  Example: Maxwell stress tensor.

The scheme can be continued to arbitrary rank-N tensors,

$$T'_{\alpha\beta\gamma\ldots}(\mathbf{x}') = \sum_{\alpha'\beta'\gamma'\ldots} a_{\alpha\alpha'}a_{\beta\beta'}a_{\gamma\gamma'}\ldots T_{\alpha'\beta'\gamma'\ldots}(\mathbf{x})$$

Spatial inversion. All orthogonal transformations with $\det(\mathcal{A}) = -1$ involve a reflection or an inversion; the latter is characterized by

$$\mathcal{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Physical quantities characterized as scalars, vectors etc. under rotations can be sub-categorized further dependent on their behavior under spatial inversion:

- True scalar = even scalar = a scalar with $S'(-\mathbf{x}) = S(\mathbf{x})$
  Examples: mass, kinetic energy, potential energy, electric potential, charge (density).
- Pseudo-scalar = odd scalar = a scalar with $S'(-\mathbf{x}) = -S(\mathbf{x})$
- Polar vector = true vector = odd vector = a vector satisfying $\mathbf{V}'(-\mathbf{x}) = -\mathbf{V}(\mathbf{x})$
  Examples: position, velocity, momentum, gradient of true scalars, force, **E**, **P**, **D**, current **j**, Poynting vector **S**.
- Axial vector = pseudo-vector = even vector = a vector satisfying $\mathbf{V}'(-\mathbf{x}) = \mathbf{V}(\mathbf{x})$
  Examples: cross products of true vectors (angular momentum, torque, etc.), **B**, **M**, **H**.

- Even tensor = a tensor that transforms as $T'_{\alpha\beta\ldots}(-\mathbf{x}) = T_{\alpha\beta\ldots}(\mathbf{x})$
- Odd tensor = a tensor that transforms as $T'_{\alpha\beta\ldots}(-\mathbf{x}) = -T_{\alpha\beta\ldots}(\mathbf{x})$
- True tensor of rank N = a tensor that transforms as $T'_{\alpha\beta\ldots}(-\mathbf{x}) = (-1)^N T_{\alpha\beta\ldots}(\mathbf{x})$
- Pseudo-tensor of rank N = a tensor that transforms as $T'_{\alpha\beta\ldots}(-\mathbf{x}) = (-1)^{N+1} T_{\alpha\beta\ldots}(\mathbf{x})$

Note. From the above it follows that the transformation law for cross products of true vectors for all orthogonal transformations (including reflection and inversion) is

$$\mathbf{A}, \mathbf{B} \quad \text{true vectors and} \quad \mathbf{C} = \mathbf{A} \times \mathbf{B} \quad \Rightarrow \quad \mathbf{C}'(\mathbf{x}') = \det(\mathcal{A})\, \mathcal{A}\mathbf{C}(\mathbf{x})$$

Time reversal. Physical quantities T are characterized as

- (Time-)even if $T'(-t) = T(t)$.
  Examples: mass, position, force, torque, potential, kinetic energy, charge (density), **E**, **P**, **D**, Maxwell stress tensor.
- (Time-)odd if $T'(-t) = -T(t)$.
  Examples: velocity, momentum, current(density), **B**, **M**, **H**, Poynting vector **S**.

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Invariance of physical laws under spatial (I) and time (T) inversions. It was demonstrated that the canonical equations for a particle in a conservative potential $V(\mathbf{x})$ are invariant under these transformations: $\frac{d}{dt}\mathbf{x} = \frac{\partial}{\partial \mathbf{p}} H(\mathbf{x}, \mathbf{p})$ is T-odd and I-odd, while $\frac{d}{dt}\mathbf{p} = -\frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \mathbf{p})$ is T-even and I-odd.
Exercise: Show this. Also, you may verify that Newton's II law is I-odd and T-even.
Note: For the physical system to be T-invariant, it is sufficient that each of its equations of motion transforms consistently as either odd or even under time reversal; the same applies for invariance under spatial inversion. ("Consistently" means that all individual terms in any given equation must transform the same way.) It is not necessary that all equations of motion transform the same way.

Transformation behavior of the Maxwell equations. It was shown that

- $\nabla \cdot \mathbf{D} = \rho$ is T-even and I-even.
- $\nabla \cdot \mathbf{B} = 0$ is T-odd and I-odd.
- $\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0$ is T-even and I-even.
- $\nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{j}$ is T-odd and I-odd.

Thus, I- and T-invariance of classical electrodynamics is established. (Invariance under rotation and charge conjugation also applies.)
Note. In case that there are "external" fields, the sources causing the external fields must also be subjected to the transformations.

Application. Elimination of fundamentally impossible terms from postulated physical laws. In a postulated physical law, a term is fundamentally impossible and must be eliminated if it does not transform in the same way as other terms for which the transformation behavior is known. (Examples: Jackson, p. 272f and homework.)

Magnetic charge. Assuming that there was magnetic charges ρ_m and currents $\mathbf{j}_m$, Maxwell's equations would be

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_e \\ \nabla \cdot \mathbf{B} &= \rho_m \\ \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= -\mathbf{j}_m \\ \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} &= \mathbf{j}_e \quad . \end{aligned} \tag{88}$$

Also, ρ_m would have to be T-odd and I-odd, and $\mathbf{j}_m$ T-even and I-even.

Maxwell's equations are invariant under a **duality transformation**

$$\begin{pmatrix} \mathbf{E} \\ Z_0\mathbf{H} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \mathbf{E}' \\ Z_0\mathbf{H}' \end{pmatrix}$$

and analogous for $\begin{pmatrix} Z_0\mathbf{D} \\ \mathbf{B} \end{pmatrix}$, $\begin{pmatrix} Z_0\rho_e \\ \rho_m \end{pmatrix}$ and $\begin{pmatrix} Z_0\mathbf{j}_e \\ \mathbf{j}_m \end{pmatrix}$. There, $\mathcal{A} = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi\cos\xi & \end{pmatrix}$ with ξ T-odd and I-odd. If there were magnetic charges and there was always a constant ratio between magnetic and electric charge, a suitable duality transformation could be made to cast Maxwell's equation into their normal, magnetic-charge-free from. Consequently, in view of the duality transformation the statement that there exist no magnetic charges can be relaxed to stating that all matter has a constant ratio of magnetic to electric charge. For convenience, we usually work in a frame in which the magnetic charge has been transformed to identical zero by the means of a suitable duality transformation.

Magnetic monopoles. In the usual frame in which "normal matter" has no magnetic charge, hypothetical magnetic charges - magnetic monopoles - would have to satisfy a quantization condition

$$g = \frac{nh}{e} \quad \text{with} \quad n \in \mathbb{N} \quad .$$

The derivation of this result, following Dirac's construction of the vector potential of a monopole, was presented and discussed in detail (Jackson, p. 278ff). The result was related to magnetic-flux quantization and the Aharonov-Bohm effect.

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Harmonic pane electromagnetic waves in source-free, loss-free linear media. Use the ansatz for the complex electric and magnetic fields,

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \mathcal{E}\exp(\mathrm{i}k\mathbf{n}\cdot\mathbf{x} - \mathrm{i}\omega t) \\ \mathbf{B}(\mathbf{x},t) &= \mathcal{B}\exp(\mathrm{i}k\mathbf{n}\cdot\mathbf{x} - \mathrm{i}\omega t) \end{aligned} \tag{89}$$

with complex constant vectors $\mathcal{E}$, $\mathcal{B}$, $\mathbf{n}$, real $k = \sqrt{\epsilon\mu}\omega = \frac{n}{c_0}\omega$, real refractive index n and vacuum velocity of light c_0. Insert into the homogeneous Helmholtz equation, the divergence Maxwell equations, and the curl equations to find, in that order,

$$\begin{aligned} \mathbf{n}\cdot\mathbf{n} &= \frac{\epsilon\mu\omega^2}{k^2} \\ \mathcal{E}\cdot\mathbf{n} &= \mathcal{B}\cdot\mathbf{n} = 0 \\ \mathcal{B} &= \sqrt{\epsilon\mu}\mathbf{n}\times\mathcal{E} \end{aligned} \tag{90}$$

In the most important cases $\mathbf{n}$ is real. In that case one finds for frequency ω, wavevector $\mathbf{k} = \mathbf{n}k$, and complex amplitude E_0 two linearly independent plane-wave solutions with orthogonal linear polarizations, namely

$$\mathcal{E} = \epsilon_1 E_0 \quad \text{and} \quad \mathcal{B} = \epsilon_2 \frac{n}{c_0} E_0$$

and

$$\mathcal{E} = \epsilon_2 E_0 \quad \text{and} \quad \mathcal{B} = -\epsilon_1 \frac{n}{c_0} E_0 \quad .$$

There, $\{\epsilon_1, \epsilon_2, \mathbf{n}\}$ form a right-handed system of real orthonormal unit vectors.

The complex pointing vector of any of these is

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2\mu}\frac{n}{c_0}|E_0|^2\mathbf{n}$$

and the complex energy density

$$u = \frac{1}{4}\left(\epsilon \mathbf{E}\cdot\mathbf{E}^* + \frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}^*\right) = \frac{\epsilon}{2}|E_0|^2$$

Since in the present case ϵ and μ are real, the physical quantities $\mathrm{Re}(\mathbf{S}) = \mathbf{S}$ and $\mathrm{Re}(u) = u$. (In certain prints of Jackson, a factor 1/2 is missing in Eqn. 7.13).

General field for given ω and $\mathbf{k} = \mathbf{n}k$: $\mathbf{E}(\mathbf{x}, t) = (\epsilon_1 E_1 + \epsilon_2 E_2)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t)$. There, $k = \sqrt{\epsilon\mu}\omega$ and E_1, E_2 complex constants.

The special cases of **linear polarization and circular polarization** have been discussed in some detail. The characterization of the polarization state using the Stokes parameters has been briefly outlined.

Reflection and refraction laws. We refer to Fig. 7.5 in Jackson. It has been shown that consideration of any of the boundary conditions on the fields, e.g. $\hat{\mathbf{z}}\cdot\mathbf{D}(\mathbf{x}, t) + \hat{\mathbf{z}}\cdot\mathbf{D}''(\mathbf{x}, t) = \hat{\mathbf{z}}\cdot\mathbf{D}'(\mathbf{x}, t) \quad \forall \quad \mathbf{x} = (x, y, z = 0)$ yields the following:

- $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$ and $\mathbf{z}$ all lie in one plane, called the plane of incidence.
- $i = r'$ (Reflection law)
- $n\sin(i) = n'\sin(r)$ (Snell's law)

The matter is discussed exhaustively in textbooks such as Griffiths, Introduction to Electrodynamics.

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Fresnel's equations. Expressions for the electric-field amplitudes of reflected and transmitted waves relative to that of the incident wave are obtained from the boundary conditions for $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{H}$ for $z = 0$, $\sigma = 0$, $\mathbf{K} = 0$, and the plane-wave relation between complex electric and magnetic-field amplitudes, $\mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}$. The resultant six equations (Eqn. 7.37 in Jackson) are not all independent, and some of them are identical zero (dependent on the polarization of the incident wave).

It is customary to quote the reflection and transmission coefficients for the electric-field amplitudes for electric fields in and transverse to the plane of incidence. In any case, the boundary conditions (Eqn. 7.37) and the known reflection and transmission angles lead to two equations for the reflection and transmission

coefficients. The result (Fresnel equations) is given in Eqns. 7.39 and 7.41 of Jackson. Note that these equations and their limits for $i = 0$ have to be viewed in context with the definition of the field directions in Fig. 7.6.. Also, note that the equations are also valid for complex refractive index and in cases with imaginary $\sin(r')$. Phase jumps of π of the reflected beam, most commonly seen upon normal reflection at an optically denser medium, were briefly mentioned.

Special phenomena. Brewster's angle, $i_B = \tan^{-1}(n'/n)$ and total internal reflection were discussed. The latter phenomenon occurs when $n' < n$ and $i > i_0 = \sin^{-1}(n'/n)$. The nature of the wave in the primed region, called evanescent wave, its energy flow, Poynting vector and penetration depth were discussed. Note that the evanescent wave can be written in the form $\mathbf{E}_{\rm ev} = \mathbf{E}\exp(\mathrm{i}k\mathbf{n}\cdot\mathbf{x} - \mathrm{i}\omega t)$ etc. with complex $\mathbf{n} = \frac{\sin(i)}{\sin(i_0)}\hat{\mathbf{x}} + \mathrm{i}\sqrt{(\frac{\sin(i)}{\sin(i_0)})^2 - 1}\hat{\mathbf{z}}$. There, the surface normal is $\hat{\mathbf{z}}$ and the plane of incidence is the xz-plane. The planes of constant phase and the wavelength of evanescent waves were explained. The phenomenon of frustrated total internal reflection was explained (see homework problem).

Classical model for the complex dielectric function $\epsilon(\omega)$. The expression

$$\epsilon_r(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j \frac{1}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j} \tag{91}$$

with particle density N, electron mass m, oscillator strength f_j of resonance j, frequency ω_j of resonance j and damping rate γ_j of resonance j was derived and discussed in detail. The typical behavior of the real and imaginary parts of $\epsilon_r(\omega)$ was sketched. It was further pointed out that the imaginary part in $\epsilon_r(\omega)$ corresponds to absorption, and the corresponding $1/e$ field decay depth was derived.

Low-frequency behavior. Various limiting cases of Eq. 91 can be considered. The case $\omega \to 0$ was treated in detail, with special emphasis on the case that there are free charge carriers in the system (that is, the lowest resonance frequency in Eq. 91 is $\omega_0 = 0$). Then, it makes sense to distinguish between bound electrons with $\omega_i \gg \omega \gtrsim 0$, $i = 1, 2, 3..$, and the free carries, the density of which is labeled N in the following. The bound charges give rise to a "background" dielectric function $\epsilon_b(\omega)$, which can be determined using Eq. 91 by summing over $i = 1, 2, 3...$ For the most usual case that $f_0 = 1$ we then find

$$\epsilon(\omega) = \epsilon_b(\omega) + \mathrm{i}\frac{Ne^2}{m\omega(\gamma_0 - \mathrm{i}\omega)} \quad .$$

It was shown that this can also be written in terms of the complex conductivity $\sigma(\omega)$, defined through the complex current density and electric field $\mathbf{j} = \sigma(\omega)\mathbf{E}$:

$$\epsilon(\omega) = \epsilon_b(\omega) + \mathrm{i}\frac{\sigma(\omega)}{\omega} \quad \text{with} \quad \sigma(\omega) = \frac{Ne^2}{m(\gamma_0 - \mathrm{i}\omega)} \quad .$$

The role of γ_0 as an effective charge-carrier collision rate was discussed.

High-frequency behavior. In plasmas and if $\omega \gg \omega_i \quad \forall i$ the dielectric function can be written in form of a dispersion relation

$$c_0 k = \sqrt{\omega^2 \epsilon_{r,b}(\omega) - \omega_p^2}$$

with plasma frequency $\omega_p = \sqrt{\frac{Ne^2}{\epsilon_0 m}}$ and vacuum velocity of light c_0. There, N is the total density of charge carriers, m their (effective) mass, and $\epsilon_{r,b}(\omega)$ an optional background relative dielectric function. The latter equals 1 in plasmas; in a metal, $\epsilon_{r,b}(\omega)$ accounts for the dielectric properties of the metal ionic cores.

The plasma dispersion relation, the plasma frequency, Langmuir plasma oscillations, and the phase and group velocities of plasma waves were discussed. The reflective property of plasmas for incident waves with $\omega < \omega_p$ was mentioned.

27 Review session 12/16/2004

Information about the exam:

- Read information posted on website.
- Recommended materials: Jackson, lecture notes, homework solutions of this class, calculator, writing materials, ruler.
- Additional allowed materials: Another textbook (Griffiths or similar), mathematical reference book.
- Not allowed materials: Worked out solutions to problems other than the homework problems of this course.
- **Exam material (general):** Whole course up to and including Chapter 7.5 plus materials of these notes.
- **Excluded materials:** Numerical methods, Sub-Chapters 6.5 and 6.6.
- **Exam problems:** Three problems. One on electrostatics (keywords: Laplace equation, variable separation, dielectrics, Green's function). One on magnetostatics (keywords: magnetic potential, vector potential, gauges, solution methods). One on plane waves (keywords: polarization, reflection and refraction, dielectric function). Other keywords: Maxwell's equations and general solutions, wave equation, Helmholtz equation, respective Green's functions, gauges of time-dependent fields.

Items discussed in some detail (some of the discussions originated in specific questions).

Problem 6.8 of the last homework; see posted solution.

Transformation properties of EM quantities and Problem 6.15 of the latest homework; see posted solution.

Spherical and cartesian multipole moments; relations between them.

Polarization states of plane waves:

In isotropic media, a plane wave of frequency ω and wave vector k has an electric field of the form

$$\mathbf{E}(\mathbf{x}) = [\hat{\epsilon_1} E_1 + \hat{\epsilon_2} E_2] \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \quad .$$

There, E_1 and E_2 are complex constants, $\hat{\epsilon_1}$ and $\hat{\epsilon_2}$ are orthogonal polarization vectors, and the time dependence has been omitted. Note that $\left\{\hat{\epsilon_1}, \hat{\epsilon_2}, \hat{\mathbf{k}}\right\}$ forms a right-handed system of orthogonal unit vectors.

Birefringence. Certain non-isotropic dielectric media have the property that for a certain propagation direction of the EM wave there exist two well-defined linear-polarization directions $\hat{\epsilon_1}$ and $\hat{\epsilon_2}$ for which the wavenumbers at fixed frequency are well defined but different (call the wavenumbers k_1 and k_2, respectively). The electric field in this situation is of the form

$$\mathbf{E}(\mathbf{x}) = [\hat{\epsilon_1} E_1 \exp(\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}) + \hat{\epsilon_2} E_2 \exp(\mathrm{i}\mathbf{k}_2\cdot\mathbf{x})] \quad .$$

There, $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel vectors with different magnitude, and the time dependence has been omitted. Typical applications are waveplates etc..

Circular polarization. Pages 299 and 230 of the textbook. For any field described by a pair of field amplitudes (E_1, E_2) there exists a well-defined pair of amplitudes (E_+, E_-) that describe the same field in the circular basis (and vice versa).

Faraday effect; birefringence for circular basis. Some dielectric media have the property that for a certain propagation direction of EM waves and fixed frequency the circular unit vectors $\hat{\epsilon}_+$ and $\hat{\epsilon}_-$ correspond to different wavenumbers k_+ and k_-. The electric field in this situation is of the form

$$\mathbf{E}(\mathbf{x}) = [\hat{\epsilon}_+ E_+ \exp(\mathrm{i}\mathbf{k}_+\cdot\mathbf{x}) + \hat{\epsilon}_- E_- \exp(\mathrm{i}\mathbf{k}_-\cdot\mathbf{x})] \quad .$$

There, $\mathbf{k}_+$ and $\mathbf{k}_-$ are parallel vectors with different magnitude, and the time dependence has been omitted.

Other matters briefly reviewed:

Chapter 6.7 (Energy- and momentum-conservation of systems involving time-dependent fields in loss-less dielectric media and particle systems).

Potentials $\Phi, \mathbf{A}$ and Coulomb- and Lorentz-gauge for time-dependent fields.

Macroscopic Maxwell's equations and applicable boundary conditions at interfaces with different electric and magnetic properties.

Eigenfunction expansion of Green's functions. Expansions of the free-space Green's fucntion of the Laplace equation, $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$.