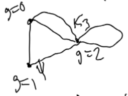
# Boundary vanishing, socle evaluations, kappa rings, and $\lambda_g$-formulas

- preliminaries: tautological rings
- $R^*(\overline{M}_{g,n})$ = subring of $A^*(\overline{M}_{g,n})$
- $\psi$ classes: $\psi_i = c_1(\mathbb{L}_i)$, $\mathbb{L}_i$ = cotangent line at $i$th marking
- $\kappa$ classes: $\kappa_m = \pi_*(\psi_{n+1}^{m+1})$ for $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$
- additive generators for $R^*(\overline{M}_{g,n})$: $\xi_{\Gamma*}$ (monomial in $\kappa$'s and $\psi$'s) for $\xi_\Gamma : \prod \overline{M}_{g_v,n_v} \to \overline{M}_{g,n}$




- $R^*(M_{g,n})$, $R^*(M^{ct}_{g,n})$ are defined by restriction

$\Gamma$ has no edges $\qquad$ $\Gamma$ is a tree

There are many nice classes in $R^*(\overline{M}_{g,n})$:

$\mathbb{E}_g$ = Hodge bundle = $\pi_* \omega_\pi$

$\lambda_i := c_i(\mathbb{E}_g) \in R^i(\overline{M}_{g,n})$.

**Conj:** For each $g \geq 2$, there exists a unique (up to scaling) nonzero class
$$\alpha_g \in R^{g-1}(\overline{M}_g) \text{ such that}$$
$$j^*\alpha_g = 0 \text{ for any } j: \overline{M}_{g_1,1} \times \overline{M}_{g_2,1} \to \overline{M}_{g_1+g_2}$$
$$(g_1+g_2=g).$$

**Thm:** Such an $\alpha_g$ exists.
(Computer: Conj is true for $g \leq 5$) $(\dim_{\mathbb{Q}} R^4(\overline{M}_5) = 165)$

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Plan: First discuss motivation/significance of $\alpha_g$, then construction at the end.

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**I: Socle evaluation maps and bridgeless curves:**

A key part of the Gorenstein conjectures for taut. ring is that they are rank 1 in the top nonvanishing degree:

$$R^{3g-3}(\overline{M}_g) \cong \mathbb{Q}$$
$$R^{2g-3}(M_g^{ct}) \cong \mathbb{Q}$$
$$R^{g-2}(M') \cong \mathbb{Q}$$

$R \quad (M_g) \cong \mathbb{Q}$

One way to describe these isoms;

$\lambda_g \in R^g(\overline{M}_g)$ has the property that it vanishes on

$\overline{M}_g \setminus M_g^{ct} = \overline{\{$  $\}}$, so can define a map

$$R^{2g-3}(M_g^{ct}) \to R^{3g-3}(\overline{M}_g)$$
$$x \longmapsto \tilde{x}\lambda_g$$

Similarly, $\lambda_{g-1}$ vanishes on $M_g^{ct} \setminus M_g$, so

$$R^{g-2}(M_g) \xrightarrow{\lambda_{g-1}} R^{2g-3}(M_g^{ct})$$

<u>Def</u>: $M_g^{bl} = \overline{M}_g \setminus \overline{\{$  $\}}$ is the moduli space of bridgeless curves.

$$\begin{tikzcd}
 & R^{3g-3}(\overline{M}_g) \cong \mathbb{Q} & \\
R^{2g-3}(M_g^{ct}) \cong \mathbb{Q} \arrow[ur, "\lambda_g", "\sim"'] & & R^{2g-2}(M_g^{bl}) \arrow[ul, "\alpha_g"'] \\
 & \arrow[ul, "\lambda_{g-1}", "\sim"'] \arrow[ur, "\lambda_g"'] &
\end{tikzcd}$$

$\alpha_g = \lambda_{g-1}$ on $M_g^{ct}$

Note: $\lambda_g \alpha_g = \lambda_g \lambda_{g-1}$

$R^{g-2}(M_g)\cong\mathbb{Q}$

Conj: $R^{2g-2}(M_g^{bl})\cong\mathbb{Q}$, $R^d(M_g^{bl})=0$ for $d>2g-2$.
(true for $g\leq 4$)

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II: Boundary splitting properties:

Question: Find families of classes $(\gamma_{g,n}\in R^*(\overline{M}_{g,n}))$

satisfying:

1) $j^*\gamma_{g,n}=\gamma_{g_1,n_1}\otimes\gamma_{g_2,n_2}$ for all
$$j:\overline{M}_{g_1,n_1}\times\overline{M}_{g_2,n_2}\to\overline{M}_{g_1+g_2,n_1+n_2-2}$$

2) $\gamma_{g,n}$ is of pure cohom. degree
$a\cdot g+b(n-2)$ for some $a,b$.

Examples:

1) $\gamma_{g,n}=\lambda_g$ (deg $g$)

2) Norbury's class $\Theta_{g,n}$ (deg $2g-2+n$)

3) $\gamma_{g,n}=\begin{cases}\alpha_g & \text{if } n=0\\ 0 & \text{else}\end{cases}$ (deg $g-1+\frac{n}{2}$)

4) Thm: There exists a family
$\alpha_{g,n} \in R^{g-1+\frac{n}{2}}(\overline{M}_{g,n})$
with this boundary splitting property
and $\alpha_{g,n} \neq 0$ for $n$ even

III: The $\kappa$ ring in compact type:

Def: $\kappa^*(M_{g,n}^{ct})$ = subring of $R^*(M_{g,n}^{ct})$
gen by $\kappa_1, \kappa_2, \kappa_3, \ldots$

Pandharipande studied this ring and determined its precise structure (for $n > 0$).

The "first" relation is part of a family of relations that looks like:

Prop: Define constants $c_i \in \mathbb{Q}$ by
$$\sum_{i \geq 0} c_i T^i = -\log\left(\sum_{i \geq 0} (2i-1)!! \, T^i\right).$$

Then $\left[e^{\sum c_i \kappa_i}\right]_{\deg d} = 0$ in $R^d(M_{g,n}^{ct})$

for $d > g-1+\frac{n}{2}$.

<u>Cor</u>: $\gamma_{g,n} := \left[ e^{\sum_i c_i \kappa_i} \right]_{\deg\, g-1+\frac{n}{2}}$ satisfies boundary splitting in $R^*(M_{g,n}^{ct})$.

<u>Philosophy</u>: taut. rels on $M_{g,n}$ or $M_{g,n}^{ct}$ should extend naturally to $\overline{M}_{g,n}$

Perhaps this $\gamma_{g,n}$ then extends to $\alpha_{g,n}$.

<u>Observation</u>: $\lambda_{g-1}|_{M_g^{ct}}$ also satisfies the $\alpha_g$-condition

<u>Conj</u>: $\lambda_{g-1}|_{M_g^{ct}} = \frac{(-1)^{g-1}}{2^{2g-1}-1} \left[ e^{\sum_i c_i \kappa_i} \right]_{\deg\, g-1}$

(True for $g \leq 5$)

In particular, conj says $\lambda_{g-1} \in \kappa^*(M_g^{ct})$.

<u>IV</u>: $\lambda_g$-formulas (and the construction of $\alpha_g$)

2 formulas for $(-1)^g \lambda_g$:

1) Use GRR to compute $ch_i(\mathbb{E})$, then exponentiate to get

$$c(\mathbb{E}^\vee) = 1-\lambda_1+\lambda_2-\cdots+(-1)^g\lambda_g.$$

2) Use the DR cycle formula to compute $DR_g(\emptyset) = (-1)^g\lambda_g$: result is naturally an impure class

$$DR_g^{total} = \sum_{d\geq 0} DR_g^{(d)}(\emptyset) \quad \text{with } DR_g^{(g)}(\emptyset) = (-1)^g\lambda_g$$

and $DR_g^{(d)}(\emptyset) = 0$ for $d>g$.

Both impure classes satisfy boundary splitting:

$$j^*c(\mathbb{E}^\vee) = \pi^*c(\mathbb{E}^\vee) \otimes \pi^*c(\mathbb{E}^\vee)$$

$$j^*DR_g^{total} = \pi^*DR_g^{total} \otimes \pi^*DR_g^{total}$$

This implies that if we take

$B_g :=$ deg $g-1$ part of (either) formula

then we have a sequence of classes

$\beta_g \in R^{g-1}(M_g)$ satisfying

$$j^* \beta_g = \beta_{g_1} \otimes (-1)^{g_2} \lambda_{g_2} + (-1)^{g_1} \lambda_{g_1} \otimes \beta_{g_2}$$

In such sequences, each $\beta_g$ is determined from the preceding $\beta_1, \ldots, \beta_{g-1}$ up to adding a class satisfying the $\alpha_g$-condition.

So if we have many such (lin. indep) sequences $(\beta_g)$, should expect to yield a construction for $\alpha_g$.

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Recall that the DR cycle formula (JPPZ) has the following form for $(-1)^g \lambda_g$:

Let $\overline{M}_g^{0/r}$ be the moduli space of (stacky) stable genus $g$ curves $C$ with a chosen $L$ with $(r \geq 1)$

$$L^{\otimes r} \cong \mathcal{O}_C.$$

It has a universal curve $\pi: \overline{\mathcal{C}}_g^{0/r} \to \overline{M}_g^{0/r}$ and a universal $r$th root line bundle $\mathcal{L}$ on $\overline{\mathcal{C}}_g^{0/r}$.

Let $\varepsilon: \overline{M}_g^{0/r} \to \overline{M}_g$.

Then the DR formula is

$$(-1)^g \lambda_g = \left[ r \, \varepsilon_* c_g(-R^* \pi_* \mathcal{L}) \right]_{r^0}.$$

Def: $\alpha_g := \left[ r^{-1} \varepsilon_* c_{g-1}(-R^* \pi_* \mathcal{L}) \right]_{r^{2g-2}}$

Equivalently,

$$\alpha_g := \lim_{r \to \infty} r^{1-2g} \varepsilon_* c_{g-1}(-R^* \pi_* \mathcal{L}).$$

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What about $\alpha_{g,n} \in R^{g-1+\frac{n}{2}}(\overline{M}_{g,n})$?

Same as above, but take

$$L^r \cong \mathcal{O}_C\left(-\frac{r}{2}\left([p_1] + \cdots + [p_n]\right)\right)$$

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Know: $\alpha_g \big|_{M_g^{ct}} = (-1)^{g-1} \lambda_{g-1} \big|_{M_g^{ct}}$

Do not know: $\alpha_{g,n} \big|_{M_g^{ct}} \in K^*(M_g^{ct})$