The Gromov-Witten theory of an elliptic curve and quasimodular forms

Aaron Pixton

May 5, 2008

Contents

Chapter 1

Introduction

Gromov-Witten theory can be viewed as the study of the moduli stack $\overline{M}_{g,n}(X,\beta)$ of stable maps from curves of genus g with n marked points to a smooth projective variety X representing a homology class $\beta \in H_2(X)$. Via obstruction theory, one can construct a virtual class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_*(\overline{M}_{g,n}(X,\beta))$, and then the Gromov-Witten invariants of X are the intersections of $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ with products of certain tautological classes in $H^*(\overline{M}_{g,n}(X,\beta))$. These classes come in two types. First, for each i there is a descendent class $\tau_k(\gamma) = c_1(\mathbb{L}_i)^k \operatorname{ev}_i^*(\gamma)$, where $\mathbb{L}_i$ is the line bundle of cotangent lines over the ith marked point and $\operatorname{ev}_i^*(\gamma)$ is the pullback of a class on X under the evaluation map at the ith marked point. Also, there are the Hodge classes $\lambda_k = c_k(\mathbb{E})$ for $1 \leq k \leq g$, where $\mathbb{E}$ is the Hodge bundle of 1-forms on the source curve.

We treat Gromov-Witten theory as a family of linear functionals

$$\langle \cdot \rangle_\beta^X : \mathcal{A}'[\hbar, \hbar^{-1}] \to \mathbb{Q},$$

where $\mathcal{A}'$ is a certain polynomial algebra parametrizing the product of insertion classes and the variable $\hbar$ serves to indicate the genus of the invariant. We often will permit the source curve to be disconnected, yielding the disconnected Gromov-Witten theory of X, which uses the notation $\langle \cdot \rangle_\beta^{X,\bullet}$. For all the relevant definitions and general background on Gromov-Witten theory, see Chapter 2.

When X is a point, Gromov-Witten theory reduces to the intersection theory of tautological classes on $\overline{M}_{g,n}$, which is a classical subject of study. When X is a curve, Gromov-Witten theory can be related to Hurwitz theory, which counts the number of branched covers of a fixed curve with a given ramification type. Because of this, there are close ties to representation theory: see the work of Okounkov and Pandharipande in [18]. When X is of higher dimension, many Gromov-Witten invariants can be interpreted enumeratively as counting the number of curves on X satisfying certain conditions. Thus in each case, these rational invariants are of great interest and are relevant to an amazing variety of areas of mathematics. There are also numerous connections to theoretical physics, which has motivated many mathematical conjectures in the subject.

One of the most important of these conjectures is due to Katz, Klemm, and Vafa ([12], or see Conjectures 1 and 2 in [14] for a mathematical treatment). Let X be a K3 surface;

that is, X is a smooth projective surface that is simply connected and has trivial canonical class. Via string-theoretic calculations, Katz, Klemm, and Vafa arrived at conjectural values for the invariants

$$\langle \lambda_g \rangle_{\beta,g}^X.$$

The KKV conjecture is described in greater detail in Chapter 5.

It is difficult to compute invariants of a K3 surface directly. The most promising approach seems to be the recent work of Maulik and Pandharipande ([13]) relating the Gromov-Witten theory of a K3 surface to that of an elliptic curve. This suggests that it would be useful to understand the elliptic curve invariants better, and that will be the primary focus of this thesis. However, Chapter 5 contains a brief exposition of this elliptic connection and performs the computations necessary to verify the KKV conjecture for $g \leq 3$.

One important feature shared by the elliptic and K3 invariants is that they can both be interpreted as Fourier coefficients of quasimodular forms. A quasimodular form can be thought of as a holomorphic function on the upper half-plane which almost satisfies the usual modular transformation property – see Chapter 4 for a definition. The algebra of quasimodular forms QM_* turns out to be the free polynomial algebra $\mathbb{Q}[E_2, E_4, E_6]$ generated by the first three Eisenstein series.

We can obtain quasimodular forms from invariants on elliptic curves by summing over different choices for the curve class β, which must be some nonnegative integer multiple of the fundamental class ω. Thus define

$$\langle \cdot \rangle^E : \mathcal{A}'[\hbar] \to \mathrm{QM}_*$$

by

$$\langle I \rangle^E := \sum_{d \geq 0} \langle I \rangle_{d\omega}^E q^d. \tag{1.1}$$

Work of Okounkov and Pandharipande ([18], [17], [19]) implies that these q-series are indeed the Fourier expansions of quasimodular forms. A similar definition can be used in the case of a K3 surface (see Chapter 5), and the quasimodularity follows from the elliptic curve case and the connection between the two proven by Maulik and Pandharipande in [13].

Although the work of Okounkov and Pandharipande provides an algorithm for computing any elliptic invariant, this algorithm is quite difficult to employ in practice. Chapter 3 is devoted to describing this algorithm and attempting to make it as nice as possible. We define "negative descendent" insertions $\tau_k(\gamma)$ with $k < 0$ and explain how to write the Hodge classes in terms of these insertions; informally (see Corollary 3.4.2), we can write

$$\mathrm{ch}_k = \frac{B_{k+1}}{(k+1)!} \sum_{i \in \mathbb{Z}} (-1)^i (\tau_i(1)\tau_{k-1-i}(\omega) + \tau_i(\alpha)\tau_{k-1-i}(\beta)), \tag{1.2}$$

where ch_k is the kth Chern character of the Hodge bundle, B_{k+1} is a Bernoulli number, and α, β are specific elements of $H^1(E)$.

Chapter 4 contains the primary results of this thesis. We attempt to fully exploit the quasimodularity of the elliptic invariants (1.1). Any quasimodular form can be uniquely written as a polynomial in the quasimodular Eisenstein series E_2 with modular coefficients; thus the differential operator $\frac{d}{dE_2}$ completely encodes the information of how far from being modular a given quasimodular form is. Surprisingly, applying this operator to an elliptic invariant is the same thing as multiplying the insertion by a formal infinite sum resembling (1.2); informally (see Theorem 4.2.1), we can say that

$$\frac{d}{dE_2} = -\frac{1}{24}\sum_{i\in\mathbb{Z}}(-1)^i\tau_i(1)\tau_{-i}(1).$$

This is an extremely useful result for two reasons. First, it can be used to prove that certain elliptic invariants which would be very difficult to compute in full are modular forms, not just quasimodular forms. Second, we can apply the Ramanujan bounds for Fourier coefficients of cusp forms to obtain asymptotics for the coefficients of these elliptic invariant quasimodular forms, which are simply the usual Gromov-Witten invariants of given degrees. The following example demonstrates both of these types of results:

Theorem 4.4.2. *The elliptic invariant $\langle\lambda_{g-1}\tau_{g-1}(\omega)\rangle$ is modular for any $g \geq 1$. Moreover, the individual coefficients satisfy*

$$\langle\lambda_{g-1}\tau_{g-1}(\omega)\rangle^E_{d\omega} = \frac{g!}{(2g)!2^{g-2}}\sigma_{2g-1}(d) + \mathbf{O}(d^{g-\frac{1}{2}+\epsilon})$$

for all $\epsilon > 0$.

The strength of these asymptotics motivates conjectures that asymptotic expansions such as those given by Theorem 4.4.2 are sometimes actually exact. In this case, this is equivalent to saying that the elliptic invariant is a scalar multiple of an Eisenstein series.

Conjecture 4.4.3. *For any $g \geq 1$,*

$$\langle\lambda_{g-1}\tau_{g-1}(\omega)\rangle^E = \frac{g!}{2^{g-1}}C_{2g}.$$

In general, all genus g elliptic invariants containing the Hodge factor λ_{g-1} seem to exhibit similar behavior.

Theorem 4.4.4. *Let $I \in \mathcal{A}'$ be any monomial in the Hodge classes and the elliptic descendent invariants and suppose that $g \geq 1$. Then there exists $C \in \mathbb{Q}$ and $e \geq 0$ such that*

$$\langle\lambda_{g-1}I\rangle^E_{g,d\omega} = Cd^e\sigma_{2g-1}(d) + \mathbf{O}(d^{e+g-\frac{1}{2}+\epsilon})$$

for any $\epsilon > 0$. If $I = \tau_{k_1}(\omega)\cdots\tau_{k_m}(\omega)\tau_{k_{m+1}+1}(1)\cdots\tau_{k_n+1}(1)$ with $k_1,\ldots,k_m,k_{m+1}+1,\ldots,k_n+1 \geq 1$ and $k_1+\ldots+k_n = g-1$, then $e = m-1$ and

$$C = \frac{(2g+n-3)!\sum_{i=1}^m(2k_i+1)}{2^{2g-2}(2g+m-2)!\prod_{i=1}^n(2k_i+1)!!},$$

where $(2k_i+1)!! = 1\cdot 3\cdots(2k_i+1)$.

We conjecture that the asymptotic expansions given by Theorem 4.4.4 are also always exact; see Conjecture 4.4.5. Thus our methods suggest tantalizingly simple formulas for many elliptic invariants. However, our methods are only sufficient to prove these results asymptotically.

Chapter 2

Gromov-Witten preliminaries

2.1 Moduli stacks of stable maps and Gromov-Witten theory

We will not go through most of the details of the definitions in this chapter. For one reference on the subject, see Part 4 of [11].

The moduli stack of stable curves of genus g with n marked points is denoted $\overline{M}_{g,n}$. A point $[C, p_1, \ldots, p_n] \in \overline{M}_{g,n}$ describes a connected nodal algebraic curve C (over $\mathbb{C}$) of arithmetic genus g with n distinct nonsingular marked points $p_1, \ldots, p_n$ and no infinitesimal automorphisms. This moduli stack is well-known to be smooth of dimension $3g - 3 + n$.

Remark. All the spaces and fundamental classes in this thesis will be defined algebraically and thus will have even real (cohomological) dimension; we will always describe them as "being of dimension" half this dimension. However, cohomology classes of odd grading will still be considered at times. We use the word "grading" rather than "dimension" to indicate this actual cohomological dimension.

Similarly, given a smooth projective variety X and a homology class $\beta \in H_2(X)$, one can define the moduli stack $\overline{M}_{g,n}(X, \beta)$ of stable maps to X from curves of genus g with n marked points. This moduli stack is less nice in general and tends not to be of pure dimension. However, there is a naturally defined "virtual fundamental class" $[\overline{M}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_{2e}(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$. This class will be of grading twice the "expected dimension" of the moduli stack, which is

$$e = (3 - \dim X)(g - 1) + n + \int_\beta c_1(X). \tag{2.1}$$

Note that this expected dimension is the same as the actual dimension when X is a point, since then $\overline{M}_{g,n}(X, 0) = \overline{M}_{g,n}$.

We will not give the obstruction-theoretic construction of the virtual class in this paper; see section 5.3 of [16] for the necessary details.

Gromov-Witten invariants are defined by intersecting $[\overline{M}_{g,n}(X, \beta)]^{\mathrm{vir}}$ with certain naturally defined cohomology classes on $\overline{M}_{g,n}(X, \beta)$. The simplest such classes are obtained by pulling back classes on X along the evaluation maps $\mathrm{ev}_1, \mathrm{ev}_2, \ldots, \mathrm{ev}_n : \overline{M}_{g,n}(X, \beta) \to X$ corresponding to the n marked points. Thus for any $\gamma_1, \gamma_2, \ldots, \gamma_n \in H^*(X)$, we have the "simple" Gromov-Witten invariant

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\, \mathrm{ev}_2^*(\gamma_2) \cdots \mathrm{ev}_n^*(\gamma_n).$$

When X is a surface, these simple Gromov-Witten invariants can usually be interpreted enumeratively as counting the number of curves on X with certain properties. We give two examples illustrating this interpretation and the complexity of the resulting counts:

Example. Let $X = \mathbb{P}^2$ and let $\beta = d[H]$ for $[H] \in H_2(\mathbb{P}^2)$ the class of a hyperplane divisor and d a positive integer. Let $n = 3d-1$. For each i, let $\gamma_i = [p] \in H^4(\mathbb{P}^2)$ be the fundamental class. Then the genus 0 invariant

$$N_d := \int_{[\overline{M}_{0,n}(\mathbb{P}^2, d[H])]^{\mathrm{vir}}} \mathrm{ev}_1^*([p]) \cdots \mathrm{ev}_{3d-1}^*([p])$$

can be interpreted enumeratively as the number of rational degree d curves on $\mathbb{P}^2$ passing through $3d-1$ generic points. It turns out that these counts satisfy the complicated recurrence

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

for $d > 1$ (see Theorem 25.1.1 of [11]).

Example. Let X be a K3 surface; that is, X is a smooth simply connected projective surface with trivial canonical class. Let $\beta \in H_2(X)$ be a primitive class in the Picard lattice with $\beta^2 = 2h - 2$ for some $h \geq 0$. The normal virtual class in this case turns out to be zero, since X can be deformed (in the symplectic category) such that it does not contain any curves. To obtain nontrivial invariants, one can define a "reduced" virtual fundamental class $[\overline{M}_{g,n}(X, \beta)]^{\mathrm{red}}$ of dimension one greater than the ordinary expected dimension $g - 1 + n$ given by (2.1). Then we can consider the reduced invariants

$$a_{gh} := \int_{[\overline{M}_{g,g}(X,\beta)]^{\mathrm{red}}} \mathrm{ev}_1^*([p]) \cdots \mathrm{ev}_g^*([p]).$$

These invariants too can be interpreted enumeratively as counting the number of genus g curves on X with a given number of nodes (corresponding to the parameter h) that pass through g generic points. Bryan and Leung ([2]) proved that these counts are given by the quasimodular generating functions

$$\sum_{h \geq 0} a_{gh} q^h = \left(\prod_{k \geq 1} (1 - q^k)^{-24} \right) \left(\sum_{n \geq 1} n \sigma_1(n) q^n \right)^g.$$

The methods used to prove this result will be explained in Chapter 5.

2.2 Cotangent lines and the Hodge bundle

The two examples at the end of the previous section demonstrate that even the simplest of Gromov-Witten invariants can be interesting and complicated. However, we will wish to consider insertions in the integrals that are more general than the simple evaluation classes pulled back from the target variety X. For this reason, we now describe certain natural vector bundles on $\overline{M}_{g,n}(X,\beta)$ and their Chern classes. These bundles will be pullbacks of bundles on the moduli space of curves $\overline{M}_{g,n}$ along the "forgetful map" $\overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$ that sends a map to the stabilization of the source curve. In other words, the following constructions will not depend on the maps to X, unlike the evaluation classes $\mathrm{ev}_i^*(\gamma)$.

First, for each marked point we can take the cotangent space to the source curve at the specified point; this describes line bundles $\mathbb{L}_i$ for $1 \leq i \leq n$, where the fiber over a moduli point $[C, p_1, \ldots, p_n]$ is T^*_{C,p_i}. The first Chern class of this bundle is denoted by $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n}(X,\beta))$.

Second, the *Hodge bundle* $\mathbb{E}$ is a rank g vector bundle with fiber $H^0(C, \omega_C)$ over the moduli point $[C, p_1, \ldots, p_n]$, where ω_C is the dualizing sheaf of C. The *Hodge classes* are then defined as the Chern classes $\lambda_k = c_k(\mathbb{E}) \in H^{2k}(\overline{M}_{g,n}(X,\beta))$ for $1 \leq k \leq g$. These classes satisfy relations proven by Mumford in [15]:

$$(1 + \lambda_1 t + \ldots + \lambda_g t^g)(1 - \lambda_1 t + \ldots + (-1)^g \lambda_g t^g) = 1. \tag{2.2}$$

Although the Hodge classes λ_k tend to appear more naturally, it is often easier to work with the Chern characters of the Hodge bundle, which are denoted by $\mathrm{ch}_k := \mathrm{ch}_k(\mathbb{E})$ and are related to the Chern classes λ_k by

$$1 + \lambda_1 t + \cdots + \lambda_g t^g = e^{\sum_{k\geq 1}(k-1)!\,\mathrm{ch}_k\, t^k}.$$

The Mumford relation (2.2) is thus equivalent to the vanishing of the even Chern characters ch_{2k}.

It is convenient to define the *Hodge algebra* to be the polynomial algebra

$$\mathcal{H} := \mathbb{Q}[\mathrm{ch}_1, \mathrm{ch}_3, \ldots]$$

in formal symbols corresponding to the odd Chern characters. This algebra is not genus-specific, so we define classes $\lambda_k \in \mathcal{H}$ by

$$\sum_{k\geq 0} \lambda_k t^k = e^{\sum_{k\geq 1}(k-1)!\,\mathrm{ch}_k\, t^k},$$

and note that these formal Hodge classes satisfy the Mumford-type relation

$$\left(\sum_{k\geq 0} \lambda_k t^k\right)\left(\sum_{k\geq 0} \lambda_k (-t)^k\right) = 1.$$

2.3 General Gromov-Witten invariants and the Virasoro conjectures

The most general type of Gromov-Witten integral that we will consider is given by intersecting a virtual fundamental class (or the reduced fundamental class in the case of a K3 surface) with a product of pullbacks of cohomology classes on X and powers of ψ-classes at the marked points, and possibly also a polynomial in the Hodge classes. In the case when there is no Hodge factor, this is known as a *descendent invariant*. The *descendent classes* are the products $\psi_i^k \operatorname{ev}_i^*(\gamma)$, for which we use the notation $\tau_k(\gamma)$ (leaving out the subscript i): we let

$$\langle \Lambda \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X := \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \Lambda \psi_1^{k_1} \cdots \psi_n^{k_n} \operatorname{ev}_1^*(\gamma_1) \cdots \operatorname{ev}_n^*(\gamma_n)$$

be the most general form for an invariant, where Λ is an element of the Hodge algebra $\mathcal{H}$, $k_1, \ldots, k_n$ are nonnegative integers, and $\gamma_1, \ldots, \gamma_n$ are cohomology classes on X.

We introduce a little terminology to describe the factors in the above invariant, called insertions. The factor $\tau_k(\gamma)$ is called a *descendent insertion* or a *descendent of* γ, and γ itself is an *evaluation class*. We refer to any element of the Hodge algebra $\mathcal{H}$ as a *Hodge insertion*, although we will primarily consider monomials in the Hodge classes λ_k or the Chern characters ch_k.

Let $\mathcal{B}$ be a basis for the cohomology of X consisting of elements of pure Hodge grading, so all $\gamma \in \mathcal{B}$ belong to $H^{p_\gamma, q_\gamma}(X)$ for some p_γ and q_γ. Let $\mathcal{A} = \mathcal{A}_*$ denote the "supercommutative" graded polynomial algebra over $\mathbb{Q}$ on the formal symbols $\tau_k(\gamma)$ for $k \geq 0$ and $\gamma \in \mathcal{B}$, where such a symbol has grading $2k - 2 + p_\gamma + q_\gamma$. Here the only relations are the supercommutativity relations

$$[a, b] := ab - (-1)^{mn} ba = 0$$

for $a \in \mathcal{A}_m$ and $b \in \mathcal{A}_n$. For odd n, we call the descendents of $\gamma \in H^n(X)$ *odd classes*; such elements are the source of noncommutativity in $\mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$ denote the graded algebra formed by adjoining the Hodge algebra to $\mathcal{A}$, where λ_k and ch_k are taken to have grading $2k$. We will also view $\mathcal{A}[\hbar, \hbar^{-1}]$ and $\mathcal{A}'[\hbar, \hbar^{-1}]$ as graded algebras, where the variable $\hbar$ has grading $2(\dim X - 3)$.

Then we can formally view Gromov-Witten theory as a collection of linear maps $\langle \cdot \rangle_\beta^X : \mathcal{A}'[\hbar, \hbar^{-1}] \to \mathbb{Q}$; here the coefficient of $\hbar^{g-1}$ is evaluated in genus g. Note that this map vanishes on all components of grading other than $x := 2 \int_\beta c_1(X)$ for dimensional reasons by (2.1). We will only consider X of dimension other than 3, so $\hbar$ will have nonzero grading. This means that there is a natural homogenization map $\mathcal{A}' \to \mathcal{A}'[\hbar, \hbar^{-1}]$ with image in $\mathcal{A}'[\hbar, \hbar^{-1}]_x$, defined by multiplying elements in $\mathcal{A}'_{x-2(\dim X - 3)n}$ by $\hbar^n$ for each $n \in \mathbb{Z}$ and annihilating elements of gradings not of this form. We will also use the notation $\langle \cdot \rangle_\beta^X : \mathcal{A}' \to \mathbb{Q}$ to indicate the composition of this homogenization map with the invariant function defined above. In other words, we will often regard the genus of an invariant as being determined by the insertion; when we want to emphasize (usually for geometric reasons) that the invariant is being computed using curves of a given genus or that the genus is nonconstant in a relation between invariants, we will either employ the variable $\hbar$ or add g as an additional subscript.

We will also want to consider the *disconnected Gromov-Witten theory* of a smooth projective variety X; this is defined in the same way as the connected theory, except that one uses moduli spaces of disconnected curves and maps from disconnected curves (recall that the arithmetic genus of the disjoint union of two curves of arithmetic genus g_1 and g_2 is $g_1 + g_2 - 1$). We indicate that an invariant is disconnected with a dot, so we also have linear maps $\langle \cdot \rangle^\bullet_\beta : \mathcal{A}'[\hbar, \hbar^{-1}] \to \mathbb{Q}$.

The Virasoro constraints were originally described as linear operators annihilating a certain generating function containing all of the descendent invariants. We take the dual viewpoint and instead view the Virasoro constraints as a certain family of linear operators $V_k : \mathcal{A}[\hbar, \hbar^{-1}] \to \mathcal{A}[\hbar, \hbar^{-1}]$; then the Virasoro conjecture (for the pair (X, β)) is that the composition of V_k with $\langle \cdot \rangle^\bullet_\beta$ is trivial. The resulting relations between invariants permit the removal of the insertions of the descendents of 1 from any descendent invariant. The Virasoro conjecture has been proven in numerous special cases; we will be primarily concerned with the case $\dim X = 1$, which was proven by Okounkov and Pandharipande in [19]. We will describe the Virasoro relations in the case $X = E$ is an elliptic curve in Section 3.3.

The work of Faber and Pandharipande in [9] can be interpreted as describing operators $Y_k : \mathcal{A}'[\hbar, \hbar^{-1}] \to \mathcal{A}'[\hbar, \hbar^{-1}]$ for odd $k \geq 1$ such that their composition with $\langle \cdot \rangle^\bullet_\beta$ is again trivial; the resulting relations permit the removal of all Hodge classes. Again, we will describe these operators in the elliptic curve setting in Section 3.4.

Chapter 3

The Gromov-Witten theory of an elliptic curve

3.1 Introduction

In a sequence of papers ([18], [17], [19]), Okounkov and Pandharipande developed an algorithm for computing any descendent invariant of a curve. This algorithm is particularly nice in the case of an elliptic curve, but it still requires several nontrivial steps and is quite difficult to use in any practical setting. We will give a slightly more elegant and usable formalization of this algorithm. To begin with, though, we must define some basic notation.

Let E be an elliptic curve; for convenience, we fix a basis $\mathcal{B} = \{1, \alpha, \beta, \omega\}$ for the cohomology of E using the Hodge grading as follows. Let $\alpha \in H^{10}(E)$ and $\beta \in H^{01}(E)$ be a symplectic basis for $H^1(E)$ (so $\alpha \cdot \beta = 1 = -\beta \cdot \alpha$), and let $\omega \in H^2(E)$ be the Poincaré dual of a point. These are (along with 1) the classes we will be pulling back from E.

The notation that we will use for elliptic curve invariants will differ slightly from that of the previous section. Since we are primarily just considering elliptic curve invariants, we omit the superscript E. Also, the possibilities for a curve class β correspond to nonnegative integers (the degree of the map), and we will sum these invariants into a generating function:

$$\langle I\hbar^{g-1}\rangle = \langle I\rangle_g := \sum_{d\geq 0} \langle I\rangle^E_{g,d\omega} q^d$$

We will often leave out the notation for the genus and view the genus g as determined by the dimension of the insertion I or by a formal factor $\hbar^g$ in the insertion. Thus we view the connected Gromov-Witten theory of an elliptic curve as a linear map

$$\langle\cdot\rangle : \mathcal{A}' \to \mathbb{Q}[[q]].$$

Example. For $d > 0$, the invariant $\langle \tau_0(\omega)\rangle^E_{1,d\omega}$ can be interpreted as counting the number of degree d maps from some elliptic curve to E. Since all such maps are isogenies, this is the same as counting index d sublattices of $\mathbb{Z}^2$, of which there are $\sigma_1(d)$. Combined with the

$d = 0$ invariant, this yields that

$$\langle \tau_0(\omega) \rangle = -\frac{1}{24} + \sum_{d>0} \sigma_1(d) q^d$$

is the weight 2 Eisenstein series. This is an example of the general fact that all elliptic curve invariants $\langle I \rangle$ are quasimodular forms.

In the disconnected case, we adopt the convention

$$\langle I \rangle_g^\bullet := \prod_{k\geq 1}(1-q^k) \sum_{d\geq 0} \langle I \rangle_{g,d\omega}^{E,\bullet} q^d.$$

Here the extra q-series factor serves the purpose of cancelling out the contribution from connected components of the domain curve which have genus 1 and no marked points. It is also the case that all disconnected invariants $\langle I \rangle^\bullet$ are quasimodular forms. In fact, one can easily express the disconnected invariants in terms of the connected invariants by means of the following basic result:

Proposition 3.1.1. *Let $I = \lambda_{l_1} \cdots \lambda_{l_n} \tau_{k_1}(\gamma_1) \cdots \tau_{k_m}(\gamma_m)$ be an arbitrary elliptic insertion. Then*

$$\langle I \rangle^\bullet = \sum_{\{1,\ldots,m\}=\sqcup_{I\in S} I} \quad \sum_{\sum_{I\in S} b_{I,j} = l_j} \quad \prod_{I\in S} \langle \lambda_{b_{I,1}} \cdots \lambda_{b_{I,n}} \prod_{i\in I} \tau_{k_i}(\gamma_i) \rangle,$$

where the first sum is taken over partitions of the set of descendent factors (into nonempty parts) and the second sum is over partitions of each Hodge index l_j into corresponding parts.

Proof. This relationship between the disconnected and connected invariants will follow from three observations.

First, the moduli space of stable maps from disconnected curves $\overline{M}^\bullet_{g,n}(E, d\omega)$ is itself disconnected, and its connected components are indexed by unordered sequences $(g_i, d_i)_{i=1}^m$ of the same length m with $g_1 + \cdots + g_m = g + m - 1$ and $d_1 + \cdots + d_m = d$, together with an assignment function from the n marked points to the m indices (which correspond to the connected components of the source curve). Each such connected component is then simply a product $\prod_{i=1}^m \overline{M}_{g_i,n_i}(E, d_i\omega)$ of connected moduli spaces, where n_i is the number of marked points assigned to part i. The virtual fundamental class also splits as a product of virtual fundamental classes in this way. The only exception to the above comments is if multiple connected components have the same genus and no marked points; in this case, one must quotient out by the group of automorphisms permuting such identical components.

Second, the Chern class λ_l on $\overline{M}^\bullet_{g,n}(E, d\omega)$ when restricted to such a connected component splits as a sum

$$\lambda_l = \sum_{l_1+\cdots+l_m=l} \prod_{i=1}^m \lambda_{l_i}^{(i)},$$

where $\lambda_{l_i}^{(i)}$ is a Chern class on the ith factor of the product.

Finally, we need the easily verified fact that the pure Hodge insertions $\langle \Lambda \rangle$ all vanish except for

$$\begin{aligned}
\langle 1 \rangle &= \sum_{d>0} q^d \sum_{f \text{ is a degree } d \text{ isogeny onto } E} \frac{1}{\mathrm{Aut}(f)} \\
&= \sum_{d>0} \frac{\sigma_1(d)}{d} q^d \\
&= \sum_{k>0} \log\left(\frac{1}{1-q^k}\right).
\end{aligned}$$

Note that this computation is the same as the one for $\langle \tau_0(\omega) \rangle$ in the example before this proposition, except that here the isogenies have nontrivial automorphisms because there are no marked points.

Now, the product decomposition of the connected components of the disconnected moduli space and the corresponding virtual classes, along with the product formula for the Hodge classes, immediately gives the correct formula for $\sum_{d\geq 0} \langle I \rangle^{\bullet}_{d\omega} q^d$ except that empty parts (meaning connected components of the source curve without marked points) are permitted and terms with duplicated empty parts must be divided by the order of the automorphism group permuting these terms. But by the above comments about pure Hodge invariants, we have that the contribution from these empty parts is precisely

$$e^{\langle 1 \rangle} = \prod_{k>0} (1-q^k)^{-1},$$

as desired. $\square$

Remark. If the Hodge class in the disconnected invariant of the previous proposition is expressed in terms of the Chern characters of the Hodge bundle rather than the Chern classes, then the analogous result holds except that the Chern characters cannot split between the different connected components.

Remark. It is slightly more complicated to describe how to express connected invariants in terms of disconnected invariants, so we do not give the general relationship here. However, most of our results, although stated for the disconnected case, will have analogues in the connected case via Proposition 3.1.1.

In Section 3.2, we describe the generating functions for the stationary elliptic invariants that Okounkov and Pandharipande obtained in [18] via Hurwitz theory. In Section 3.3, we describe the Virasoro constraints that enable the reduction of any descendent invariant to stationary invariants. In Section 3.4, we describe the relations determined by Faber and Pandharipande in [9] that permit the removal of Hodge classes from Gromov-Witten invariants. At each step, we provide a somewhat nicer formalization of this algorithm, mainly consisting of replacing differentiation operators with "negative descendents" $\tau_k(\gamma)$ with $k < 0$. Finally, in Section 3.5 we consider the example of the invariants $\langle \mathrm{ch}_k \, \tau_l(\omega) \rangle$ and demonstrate our methods.

3.2 Stationary invariants

The invariants of the form

$$\langle \tau_{k_1}(\omega)\cdots\tau_{k_n}(\omega)\rangle^\bullet$$

are known as *stationary invariants* and were computed by Pandharipande and Okounkov ([18]), who related them to Hurwitz numbers counting branched covers of E, which had previously been computed using representation theory.

Let

$$\Theta(z) = (e^{\frac{z}{2}} - e^{-\frac{z}{2}})\prod_{k\geq 1}\frac{(1-q^k e^z)(1-q^k e^{-z})}{(1-q^k)^2} = \frac{\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{(n+\frac{1}{2})z}}{\sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{1}{2}(n+\frac{1}{2})^2}}$$

be the (essentially unique) genus 1 theta function, normalized for later convenience such that $\Theta'(0)=1$. Then for each nonnegative integer n, the *n-point correlation function F_n* is defined by the expression

$$F_n(z_1,\ldots,z_n) := \sum_{\sigma\in\mathfrak{S}_n}\sigma\left(\frac{\det\left(\frac{\Theta^{(j-i+1)}(z_1+\cdots+z_{n-j})}{(j-i+1)!}\right)_{1\leq i,j\leq n}}{\Theta(z_1)\Theta(z_1+z_2)\cdots\Theta(z_1+\cdots+z_n)}\right). \tag{3.1}$$

Here the sum varies over all $n!$ permutations of the indices $1,\ldots,n$ and $\frac{1}{(-k)!}$ is taken to be zero for $k>0$; also, $F_0=1$.

Although it is not obvious from this definition, F_n can be expanded as a Laurent series in the z_i, so we also formally view F_n as an element of $\mathbb{Q}((z_1,\ldots,z_n))[[q]]$. This is more transparent in the following interpretation of the n-point correlation function, which is of great use (and justifies the name).

Theorem 3.2.1. *(Theorem 0.5 of [1]) For any $n\geq 1$, we have (in the appropriate region of convergence)*

$$\frac{\sum_\mu\prod_{k=1}^n\sum_{i\geq 1}e^{(\mu_i-i+\frac{1}{2})z_k}q^{|\mu|}}{\sum_\mu q^{|\mu|}} = F_n(z_1,\ldots,z_n),$$

where the sums are over all partitions $\mu=(\mu_1\geq\mu_2\geq\cdots=0)$ and $|\mu|=\sum_i\mu_i$ is the size of μ.

Since $\sum_{i\geq 1}e^{(\mu_i-i+\frac{1}{2})z}$ has a meromorphic expansion around 0 of the form $\frac{1}{z}+\mathbf{O}(z)$, standard analytic continuation arguments give that the above theorem implies that $F_n(z_1,\ldots,z_n)$ can be viewed as an element of $\frac{1}{z_1\cdots z_n}\mathbb{Q}[[z_1,\ldots,z_n,q]]$.

Okounkov and Pandharipande proved the following theorem (Theorem 5 of [18]):

Theorem 3.2.2. *The n-point stationary invariants of an elliptic curve are the coefficients of F_n. More precisely, for any nonnegative integers $k_1,\ldots,k_n$, we have*

$$\langle\tau_{k_1}(\omega)\cdots\tau_{k_n}(\omega)\rangle^\bullet = [z_1^{k_1+1}\cdots z_n^{k_n+1}]F_n(z_1,\ldots,z_n).$$

One consequence of this theorem is that all stationary invariants are quasimodular forms. The reason for this is that $\Theta(z)$ can be rewritten in terms of certain normalized Eisenstein series C_{2k} (see Chapter 4):

$$\Theta(z) = ze^{-\sum_{k\geq 1} C_{2k} z^{2k}}.$$

We will see in the next two sections that any elliptic curve invariant can be reduced to the stationary case, so this will imply that all invariants of an elliptic curve are quasimodular forms.

Because the generating function F_n contains some terms (with negative powers of some z_i) whose coefficients are not described by the above result, it turns out to be natural to make the following definition (first made in [18]):

The formal insertion $\tau_{-2}(\omega)$ (of grading -4) can be removed from a stationary disconnected invariant (increasing the genus by 1). In other words,

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \tau_{-2}(\omega) \hbar^{g-1} \rangle^{\bullet} = \langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \hbar^{g} \rangle^{\bullet}.$$

If we employ Proposition 3.1.1 to relate the connected and disconnected cases, then we can see that it should be equivalent to say that all stationary connected invariants containing $\tau_{-2}(\omega)$ vanish except for $\langle \tau_{-2}(\omega) \hbar^{-1} \rangle = 1$.

If we set all stationary invariants containing some formal insertion $\tau_k(\omega)$ with $k < 0, k \neq -2$ equal to zero, then we can write:

Proposition 3.2.3. *The n-point function F_n satisfies the identity*

$$F_n(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n \in \mathbb{Z}} \langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle^{\bullet} z_1^{k_1+1} \cdots z_n^{k_n+1}.$$

Proof. By Theorem 3.2.2, the coefficients of $z_1^{k_1+1} \cdots z_n^{k_n+1}$ on each side are the same when all the k_i are positive. Expanding the "average over partitions" expression for F_n provided by Theorem 3.2.1 as a Laurent series about 0, we can see that the only other monomials (in the z_i) that occur in the expansion of F_n have all of the k_i positive except for some which are -2. By the definition of the formal stationary invariants, it only remains to check that $[z_1^{k_1+1} \cdots z_{n-1}^{k_{n-1}+1} z_n^{-1}]F_n = [z_1^{k_1+1} \cdots z_{n-1}^{k_{n-1}+1}]F_{n-1}$, and this again follows from Theorem 3.2.1. $\square$

3.3 Non-stationary descendents

Okounkov and Pandharipande ([19]) described how to evaluate all descendent invariants by a two-step reduction to the case of stationary invariants. First, insertions of the form $\tau_k(1)$ are removed via the Virasoro constraints, and then the odd classes $\tau_k(\alpha), \tau_k(\beta)$ can be removed from the resulting invariants (which just involve stationary classes and odd classes).

Okounkov and Pandharipande present this constraints in [19] as linear operators that annihilate the exponential generating function for descendent invariants; as explained earlier, we take the dual viewpoint and view these operators as maps $\mathcal{A} \to \mathcal{A}$ which annihilate the Gromov-Witten map $\mathcal{A} \to \mathbb{Q}[[q]]$ by precomposition.

Recall that $\mathcal{B} = \{1, \alpha, \beta, \omega\}$ is our chosen basis for the cohomology of E and $\gamma \in H^{p_\gamma q_\gamma}(E)$ for $\gamma \in \mathcal{B}$.

For $k \geq 0$, the Virasoro operator $V_k : \mathcal{A} \to \mathcal{A}$ is given by

$$V_k = -\tau_{k+1}(1) + \sum_{\substack{l \geq 0 \\ \gamma \in \mathcal{B}}} \binom{k+l+p_\gamma}{k+1} \tau_{k+l}(\gamma) \frac{d}{d\tau_l(\gamma)}.$$

For $k = -1$, an additional term is needed to account for the fact that $k+l$ is negative when $l = 0$:

$$V_{-1} = -\tau_0(1) + \sum_{\substack{l \geq 1 \\ \gamma \in \mathcal{B}}} \tau_{l-1}(\gamma) \frac{d}{d\tau_l(\gamma)} + \hbar \left(\frac{d}{d\tau_0(1)} \frac{d}{d\tau_0(\omega)} + \frac{d}{d\tau_0(\beta)} \frac{d}{d\tau_0(\alpha)} \right).$$

Here the factor of $\hbar$ in the final term indicates that removing $\tau_0(1)$ can actually cause the genus of the invariant to increase.

Okounkov and Pandharipande also considered a similarly defined family of pairs of operators $W_k, \overline{W}_k : \mathcal{A} \to \mathcal{A}$ for $k \geq -1$, given by

$$W_k = -\tau_{k+1}(\beta) + \sum_{\substack{l \geq 0 \\ \gamma \in \mathcal{B}}} \binom{k+l+p_\gamma}{k+1} \tau_{k+l}(\beta \cup \gamma) \frac{d}{d\tau_l(\gamma)}$$

and

$$\overline{W}_k = -\tau_{k+1}(\alpha) + \sum_{\substack{l \geq 0 \\ \gamma \in \mathcal{B}}} \binom{k+l+q_\gamma}{k+1} \tau_{k+l}(\alpha \cup \gamma) \frac{d}{d\tau_l(\gamma)}.$$

These operators provide many useful relations between disconnected invariants by the following result (Theorems 3 and 4 in [19]):

Theorem 3.3.1. *For all $k \geq -1$ and $I \in \mathcal{A}$, $\langle V_k I \rangle^\bullet = \langle W_k I \rangle^\bullet = \langle \overline{W}_k I \rangle^\bullet = 0$.*

Let $\mathcal{S}$ be the subalgebra of $\mathcal{A}$ generated by the stationary insertions. Then it is easily seen that $\mathcal{S}$ and the images of the V_k, W_k, and $\overline{W}_k$ operators span $\mathcal{A}$, so Theorem 3.2.2 together with Theorem 3.3.1 determine all descendent invariants of an elliptic curve.

In practice, though, it is usually easier to apply the following result than to remove the non-stationary insertions one by one using the above operators.

Proposition 3.3.2. *For any $k_1, \ldots, k_n \geq 0$ and $\gamma_1, \ldots, \gamma_n \in \mathcal{B}$, we have*

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^\bullet = \left\langle \sum_{\substack{\{1,\ldots,n\} = \sqcup_{I \in S} I \\ \cup_{i \in I} \gamma_i = \pm\omega}} \operatorname{sgn}(S) \prod_{I = \{i_1, \ldots, i_m\} \in S} \binom{k_{i_1} + \cdots + k_{i_m}}{k_{i_1}, \ldots, k_{i_m}} \tau_{1 + \sum_{i \in I}(k_i - 1)}(\omega) \right\rangle^\bullet$$

and identically for the connected invariants. The sum is over all partitions of the insertions such that the classes in each part pulled back from E have product equal to $\pm\omega$. Thus this is the empty sum unless an equal number of the γ_i are α and β, and in this case the sign factor $\operatorname{sgn}(S)$ *is defined to be the sign of the matching between the α-descendents and the β-descendents.*

Proof. We prove this result in the disconnected case; the connected case will then follow formally using Proposition 3.1.1.

Let $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle'$ denote the expression on the right side of the proposed identity, and extend linearly to get another linear map $\langle\cdot\rangle' : \mathcal{A} \to \mathbb{Q}[[q]]$. This map trivially agrees with $\langle\cdot\rangle^\bullet$ on the stationary subalgebra $\mathcal{S}$, so it will suffice to show that it satisfies the relations of Theorem 3.3.1. This amounts to the identity

$$\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} = \binom{k_1+k_n}{k_1}\binom{k_1+\cdots+k_n-1}{k_1+k_n-1, k_2,\ldots,k_{n-1}} + \sum_{i=2}^{n-1}\binom{k_1+k_i-1}{k_1}\binom{k_1+\cdots+k_n-1}{k_1+k_i-1,k_2,\ldots,\widehat{k_i},\ldots,k_n},$$

which follows from inspection. □

Definition. The insertion on the right side of Proposition 3.3.2 is called the *reduction* of the insertion I on the left side, and it is denoted I^{red}.

Thus Proposition 3.3.2 simply states that $\langle I\rangle^\bullet = \langle I^{\mathrm{red}}\rangle^\bullet$.

We would like to use Proposition 3.3.2 to extend the above theory to handle arbitrary formal descendent insertions; in other words, we want to consider $\tau_k(\gamma)$ for arbitrary $k\in\mathbb{Z}$. We already defined arbitrary stationary invariants in the preceding section, so define

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet$$

for $k_1,\ldots,k_n\in\mathbb{Z}$ and $\gamma_1,\ldots,\gamma_n\in\mathcal{B}$ by the statement of Proposition 3.3.2. To handle the negative indices, we introduce the convention that a multinomial coefficient $\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n}$ is zero whenever some k_i is negative for $i>1$. If k_1 is the only negative term, the coefficient is still nonzero as long as $k_1+\cdots+k_n$ is also negative, and in this case it is given by analytic continuation of the corresponding ratio of gamma functions. Of course, this convention means that the ordering of these k_i matters, and we order the multinomial coefficients appearing in our definition as they were ordered in the original insertion.

As a consequence, the negative descendents do not necessarily supercommute with the other descendents. In order to determine the supercommutators, consider that the only possible parts involving a negative-subscript descendent that contribute nontrivially to the resulting stationary invariant are $\tau_{-2}(\omega)$ and $\tau_{-k-1}(\gamma_1)\tau_k(\gamma_2)$ for $k\geq 0$ and $\gamma_1\cup\gamma_2=\pm\omega$. We then can easily see that

$$[\tau_{k_1}(\gamma_1),\tau_{k_2}(\gamma_2)] = \delta_{k_1+k_2,-1}(-1)^{k_2}(\gamma_1\cdot\gamma_2), \tag{3.2}$$

in the sense that invariants are preserved by supercommuting insertions in this way.

In fact, the same argument shows that the following simpler definition for the negative descendent symbols leads to the same invariants as the definition above.

Definition. For $k \geq 0$ and $\gamma_1, \gamma_2 \in \mathcal{B}$ such that $(k, \gamma_1) \neq (1, \omega)$ and $\gamma_1 \cup \gamma_2 = \epsilon\omega$ for $\epsilon = \pm 1$, let the formal descendent symbol $\tau_{-k-1}(\gamma_1)$ be defined as a linear operator on $\mathcal{A}$ by

$$\tau_{-k-1}(\gamma_1) = \left((-1)^k \epsilon \frac{d}{d\tau_k(\gamma_2)}\right) \hbar.$$

Also define

$$\tau_{-2}(\omega) = \left(1 - \frac{d}{d\tau_1(1)}\right) \hbar.$$

Disconnected descendent invariants involving these symbols are defined by replacing them by the corresponding operators and applying the operators to obtain an element of $\mathcal{A}$.

Remark. A corresponding definition can be made in the connected setting such that the relationship described in Proposition 3.1.1 between the disconnected and connected invariants still holds. However, we do not give the details here because we will be primarily concerned with disconnected invariants.

Thus the negative descendents are really just new names for the differentiation operators (with genus change factors $\hbar$). The advantage of this definition is twofold. First, operators such as the Virasoro operators V_k become simpler and more elegant to write down and use. We can write

$$\begin{aligned}
V_k &= \sum_{i \in \mathbb{Z}} (-1)^i \binom{i}{k+1} (\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta)) \\
W_k &= \sum_{i \in \mathbb{Z}} (-1)^i \binom{i}{k+1} \tau_i(\beta)\tau_{k-1-i}(\omega) \\
\overline{W}_k &= \sum_{i \in \mathbb{Z}} (-1)^i \binom{i}{k+1} \tau_i(\alpha)\tau_{k-1-i}(\omega).
\end{aligned}$$

Second, as a consequence of the fact that the negative descendent invariants can still be computed by means of Proposition 3.3.2, the negative descendents can usually be treated analogously to the normal descendent insertions. For instance, the operators V_k, W_k, and $\overline{W}_k$ are now naturally defined for all $k \in \mathbb{Z}$ (here the binomial coefficients should be defined using gamma functions), not just for $k \geq -1$, and it is easily checked that these operators still satisfy Theorem 3.3.1. In other words, the negative descendents can still be removed using Virasoro constraints.

3.4 Chern characters

Mumford ([15]) used the Grothendieck-Riemann-Roch formula to express the Chern characters ch_k of the Hodge bundle $\mathbb{E}$ on $\overline{M}_g$ in terms of certain tautological classes. Faber and

Pandharipande ([9]) then used this result to describe how to remove insertions of Chern characters of the Hodge bundle from arbitrary Gromov-Witten invariants. Recall that the conversion between Chern classes and Chern characters is quite annoying; the relation between the two is

$$1 + \lambda_1 t + \cdots + \lambda_g t^g = e^{\sum_{k\geq 1}(k-1)!\,\mathrm{ch}_k\, t^k}.$$

This has the consequence that the following results describing how to remove the ch_k insertions are not as useful as one might wish for removing the λ_k insertions.

Recall that $\mathcal{A}'$ is the algebra of insertions of descendent or Hodge type. The operators $Y_k : \mathcal{A}'[\hbar, \hbar^{-1}] \to \mathcal{A}'[\hbar, \hbar^{-1}]$ for $k \geq 1$ are defined by

$$Y_k = -\mathrm{ch}_k + \left(\frac{B_{k+1}}{(k+1)!}\right) \cdot \left(\tau_{k+1}(1) - \sum_{\substack{l\geq 0\\ \gamma\in\mathcal{B}}} \tau_{k+l}(\gamma)\frac{d}{d\tau_l(\gamma)} + \hbar^{-1}\sum_{i=0}^{k-1}(-1)^i(\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta))\right).$$

Here B_{k+1} is a Bernoulli number, which are defined by the generating function $\sum_{n\geq 0} B_n z^n = \frac{z}{e^z-1}$. Note that $Y_k = 0$ for even k, since then ch_k and B_{k+1} both vanish.

Then the result of Faber and Pandharipande (Proposition 2 of [9]) can be stated as follows:

Proposition 3.4.1. *For any insertion $I \in \mathcal{A}'[\hbar, \hbar^{-1}]$ and $k \geq 1$, $\langle Y_k I\rangle^\bullet = 0$.*

Using the formalization of the previous section, we can replace the differentiation operators in Y_k by negative descendents to write $Y_k = -\mathrm{ch}_k + \frac{B_{k+1}}{(k+1)!}\varpi_k$, where

$$\varpi_k = \hbar^{-1}\sum_{i\in\mathbb{Z}}(-1)^i(\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta)).$$

Since $\mathcal{A}'$ is spanned by $\mathcal{A}$ and the image of the Y_k operators, Proposition 3.4.1 allows us to write any Gromov-Witten invariant in terms of descendent invariants:

Corollary 3.4.2. *For any odd $k_1, \ldots, k_m$ and descendent insertion $I \in \mathcal{A}$,*

$$\langle \mathrm{ch}_{k_1}\cdots\mathrm{ch}_{k_m} I\rangle^\bullet = \left\langle \prod_{i=1}^m \frac{B_{k_i+1}}{(k_i+1)!}\varpi_{k_i} I\right\rangle^\bullet.$$

Note that ϖ_k is nontrivial for even k, unlike ch_k; this will be relevant when constructing generating functions in the next section, for which it is more natural to remove the Bernoulli factors.

It is important to note that ϖ_k does not commute with the descendent classes, even though ch_k does commute. In other words, the informally stated identity

$$\mathrm{ch}_k = \frac{B_{k+1}}{(k+1)!}\varpi_k$$

should only be treated as an identity of operators (acting on the left). From the supercommutation relations (3.2), we can easily compute that (for odd k)

$$[\tau_i(\gamma), \varpi_k] = \tau_{i+k}(\gamma).$$

We will tend to try to avoid the confusion between $\mathrm{ch}_k \in \mathcal{A}'$ and $\varpi_k \in \mathrm{End}(\mathcal{A})$ by performing computations in $\mathcal{A}$ rather than $\mathcal{A}'$ whenever possible.

3.5 An example

As an example of the formalization in this chapter, we compute $\langle \mathrm{ch}_k \, \tau_l(\omega)\rangle = \langle \mathrm{ch}_k \, \tau_l(\omega)\rangle^\bullet$ for all odd positive integers k and l. We begin by noting that the formal invariants $\langle \varpi_k \tau_l(\omega)\rangle$ (now without a parity restriction on the indices) have a rather simple generating function: we have

$$\begin{aligned}
\langle \varpi_k \tau_l(\omega)\rangle &= \langle \varpi_k \tau_l(\omega)\rangle^\bullet \\
&= \sum_{i\in\mathbb{Z}} (-1)^i \binom{i+l}{i,l} \langle \tau_{k-1-i}(\omega)\tau_{i+l-1}(\omega)\rangle^\bullet \\
&= \sum_{i\in\mathbb{Z}} (-1)^i \binom{i+l}{i,l} [z^{k-i}x^{i+l}] F_2(z,x) \\
&= [z^k x^l] F_2(z, x-z),
\end{aligned}$$

where the final Laurent series is expanded in x/z and z, so $\frac{1}{x-z} = -\frac{1}{z} - \frac{x}{z^2} - \cdots$.

To compute these coefficients, it is useful to recall that

$$\frac{1}{\Theta(z)} = \sum_k \langle \tau_k(\omega)\rangle z^{k+1}$$

and

$$\frac{\Theta'(z)}{\Theta(z)} = \frac{1}{z} - \sum_k k C_k z^{k-1},$$

where the $C_k \in \mathbb{Q}[[q]]$ are appropriately normalized Eisenstein series (see Chapter 4).

Using the expression (3.1) for F_2, we have that

$$\begin{aligned}
F_2(z, x-z) &= \frac{\Theta'(z)}{\Theta(z)\Theta(x)} + \frac{\Theta'(x-z)}{\Theta(x-z)\Theta(x)} \\
&= \left(\frac{1}{z} + \frac{1}{x-z}\right) \sum_j \langle \tau_j(\omega)\rangle x^{j+1} - \sum_{i,j} i C_i \langle \tau_j(\omega)\rangle x^{j+1} (z^{i-1} + (x-z)^{i-1}).
\end{aligned}$$

Note that the first term does not contribute to the coefficients of interest to us, since the exponent of z is negative there. Taking the coefficient of $z^k x^l$ in the second term yields the following result:

Proposition 3.5.1. *The identity*

$$\langle \mathrm{ch}_k\, \tau_l(\omega)\rangle = \frac{B_{k+1}}{k!}\sum_{i=1}^{\frac{l+1}{2}} \binom{2i+k+1}{k+1} C_{2i+k+1}\langle \tau_{l-1-2i}(\omega)\rangle$$

holds for all odd positive integers k and l.

In particular, setting $k = 2g-3, l = 1$ in Proposition 3.5.1 yields that

$$\langle \mathrm{ch}_{2g-3}\, \tau_1(\omega)\rangle = \frac{B_{2g-2}}{(2g-3)!}\binom{2g}{2} C_{2g}. \tag{3.3}$$

This is actually a modular form, since it is just a renormalized Eisenstein series. In the next chapter, we will build a general framework for explaining why elliptic invariants such as this one are modular (and not just quasimodular). In addition, we will see that the fact that this invariant is an Eisenstein series is a special case of a significantly more general conjecture (see Conjecture 4.4.5 and Proposition 4.4.6).

In general, we can apply the same procedure used above to interpret $\langle \varpi_{k_1}\cdots\varpi_{k_n}\tau_l(\omega)\rangle$ as the coefficient of $z_1^{k_1}\cdots z_n^{k_n}x^l$ in the Laurent expansion of a linear combination of $(n+1)^{n-1}$ linear reparametrizations of the $(n+1)$ point correlation function F_{n+1}. For example,

$$\begin{aligned}\langle \varpi_{k_1}\varpi_{k_2}\tau_l(\omega)\rangle =& [z_1^{k_1}z_2^{k_2}x^l]((x-z_1-z_2)F_3(z_1,z_2,x-z_1-z_2) + z_1F_3(z_1-z_2,z_2,x-z_1)\\ &+ z_2F_3(z_1,z_2-z_1,x-z_2)).\end{aligned}$$

Chapter 4

Modularity and asymptotic expansions

4.1 Quasimodular forms

A *modular form* of even weight k (on $\mathrm{SL}_2(\mathbb{Z})$) is a holomorphic function f on the upper half-plane (including $i\infty$) satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \tag{4.1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

One important family of examples of modular forms is the *Eisenstein series* E_k for even $k \geq 4$, which is a weight k modular form with Fourier expansion

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \tag{4.2}$$

where $q = e^{2\pi i z}$ and B_k is a Bernoulli number.

It is well-known that the graded algebra generated by the modular forms on $\mathrm{SL}_2(\mathbb{Z})$ is freely generated by E_4 and E_6. In particular, all of the higher-weight Eisenstein series are polynomials in E_4 and E_6.

One can also define a "weight 2" Eisenstein series E_2 by (4.2). This function is not quite modular, but $E_2 - \frac{3}{\pi y}$ does satisfy (4.1) in weight $k = 2$, where $y = \mathrm{Im}(z)$. Because of this, we call E_2 a quasimodular form. In general, we say that an *almost holomorphic modular form* (of weight k) is a polynomial in $\frac{1}{y}$ with q-series coefficients that satisfies (4.1), and a *quasimodular form* (of weight k) is the constant term of such a polynomial. It turns out (see Proposition 3.5 of [1]) that the graded algebra of quasimodular forms QM_* is generated freely by the first three Eisenstein series, so $\mathrm{QM}_* = \mathbb{Q}[E_2, E_4, E_6]$. Thus one way of thinking about quasimodular forms is as polynomials in E_2 with modular coefficients. We will view quasimodular forms as interchangeable with their Fourier expansion q-series, so QM_* is a subalgebra of $\mathbb{Q}[[q]]$.

It is convenient to consider a normalization of the Eisenstein series different from that of (4.2). For each $k \geq 2$, let

$$C_k := -\frac{B_k}{k \cdot k!} E_k = -\frac{B_k}{k \cdot k!} + \frac{2}{k!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

The reason for this choice of normalization is the identity

$$\Theta(z) = z e^{-\sum_{k \geq 1} C_{2k} z^{2k}} \tag{4.3}$$

for the theta function discussed in Section 3.2.

As a consequence of this identity, the one-point stationary invariants of an elliptic curve satisfy

$$\sum_{g \geq 0} \langle \tau_{2g-2}(\omega) \rangle z^{2g} = e^{\sum_{k \geq 1} C_{2k} z^{2k}}$$

and thus can be simply expressed in terms of the Eisenstein series. This fact was used in Section 3.5.

As the previous example might indicate, we will be primarily interested in quasimodular forms simply as nice q-series which frequently show up as generating functions. As such, it is important to note that the "theta differentiation operator" $\theta := q \frac{d}{dq}$ (not to be confused with the theta function $\Theta(z)$) restricts to a derivation on $\mathrm{QM}_* = \mathbb{Q}[C_2, C_4, C_6]$, since there are the following basic identities:

$$\begin{aligned} \theta C_2 &= -2C_2^2 + 10C_4 \\ \theta C_4 &= -8C_2C_4 + 21C_6 \\ \theta C_6 &= -12C_2C_6 + \frac{160}{7} C_4^2. \end{aligned} \tag{4.4}$$

4.2 The derivation ∂

In this section, we discuss another important derivation on the algebra of quasimodular forms $\mathrm{QM}_* = \mathbb{Q}[C_2, C_4, C_6]$, namely $\partial := \frac{d}{dC_2}$. Since the subalgebra of modular forms is $\mathbb{Q}[C_4, C_6]$, this derivation has the nice property that $\partial f = 0$ if and only if f is modular. We will later use this characterization to prove that certain elliptic curve invariants are actually modular forms.

Since θ increases weight by 2 by (4.4) and ∂ decreases weight by 2, it is natural to consider the commutator of these two derivations. Straightforward computation using (4.4) yields that

$$[\theta, \partial] = 2k \tag{4.5}$$

as an operator on QM_k.

The derivation ∂ interacts especially nicely with the elliptic curve invariant formalism of the previous chapter. Define an infinite sum of formal descendent insertions by

$$\delta := \hbar^{-1} \sum_{i \in \mathbb{Z}} (-1)^i \tau_i(1) \tau_{-i}(1).$$

Then we have the following result:

Theorem 4.2.1. $\partial \langle I \rangle^\bullet = \langle \delta I \rangle^\bullet$ *for any insertion* $I \in \mathcal{A}'$.

We will need to prove a few preliminary results before approaching this theorem. First, the n-point correlation function F_n interacts very nicely with the differentiation operator ∂:

Lemma 4.2.2. *For any* $n \geq 1$*, we have the identity of formal Laurent series*

$$\begin{aligned} \partial F_n(z_1, \ldots, z_n) &= (z_1 + \cdots + z_n)^2 F_n(z_1, \ldots, z_n) \\ &\quad - 2 \sum_{1 \leq i < j \leq n} (z_i + z_j) F_{n-1}(z_i + z_j, z_1, \ldots, \widehat{z_i}, \ldots, \widehat{z_j}, \ldots, z_n) \end{aligned}$$

in $\mathrm{QM}_*((z_1, \ldots, z_n))$.

Proof. It will be convenient to let

$$M_n = M_n(z_1, \ldots, z_n) = \left(\frac{\Theta^{(j-i+1)}(z_1 + \cdots + z_{n-j})}{(j-i+1)!} \right)_{1 \leq i,j \leq n}$$

be the matrix appearing in the definition of F_n (see (3.1)). Let m_n^{ij} be the entries of M_n, so m_n^{ij} vanishes for $i > j+1$.

From (4.3), we have that $\partial \Theta(z) = -z^2 \Theta(z)$. Differentiating with respect to z yields

$$\partial \frac{\Theta^{(m)}(z)}{m!} = -z^2 \frac{\Theta^{(m)}(z)}{m!} - 2z \frac{\Theta^{(m-1)}(z)}{(m-1)!} - \frac{\Theta^{(m-2)}}{(m-2)!}.$$

Applying ∂ to (3.1) and using multilinearity of the determinant then gives

$$\begin{aligned} \partial F_n(z_1, \ldots, z_n) &= (z_1 + \cdots + z_n)^2 F_n(z_1, \ldots, z_n) \\ &\quad - 2 \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\sum_{1 \leq j \leq n} (z_1 + \cdots + z_{n-j}) \frac{\det M_n^j}{\Theta(z_1) \cdots \Theta(z_1 + \cdots + z_n)} \right) \\ &\quad - \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\sum_{1 \leq i \leq n} \frac{\det M_n^{(i)}}{\Theta(z_1) \cdots \Theta(z_1 + \cdots + z_n)} \right), \end{aligned}$$

where M_n^j is the matrix formed by shifting the jth column of M up by one entry and $M_n^{(i)}$ is formed by replacing the ith row of M with the $(i+2)$th row of M (or by a row of zeroes if $i + 2 > n$).

The first of these three terms is the first term of the desired expression, and the last of the three terms vanishes because $M_n^{(i)}$ is clearly always singular. Thus it remains to show that the second term is as desired.

We claim that in fact

$$\sum_{j=1}^{n}(z_1+\cdots+z_{n-j})\det M_n^j(z_1,\ldots,z_n) = \\ \sum_{l=1}^{n-1} z_l\Theta(z_l+\cdots+z_l)\det M_{n-1}(z_1,\ldots,z_{l-1},z_l+z_{l+1},z_{l+2},\ldots,z_n). \tag{4.6}$$

To see this, expand the left-hand side of (4.6) as a polynomial in the nonzero entries of M. The typical term in this expansion is

$$\pm(z_1+\cdots+z_{n-j})m_n^{i_1 1}\cdots m_n^{i_n n},$$

where $(i_1,i_2,\ldots,i_j+1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$ and the sign factor is the sign of this permutation.

For fixed $i_1,\ldots,i_n$, there are either 0 or 2 possible choices for j that yield such a term. If $j<j'$ are the 2 choices, then one must have $i_j=i_{j'}$. From the vanishing of m_n^{ij} for $i>j+1$, we can additionally determine that $i_k=k+1$ for $j\le k<j'$. For each such k, $m_n^{i_k k}=\Theta(z_1+\cdots+z_{n-k})$, so we can find this monomial in the m_n^{ij} on the right-hand side of (4.6) with $l=n-k$. Comparing the coefficients of this monomial on each side (and taking signs into account), we can check that $(z_1+\cdots+z_{n-j})-(z_1+\cdots+z_{n-j'})=\sum_{j\le k<j'} z_{n-k}$, so identity (4.6) holds.

Dividing (4.6) by $\Theta(z_1)\cdots\Theta(z_1+\cdots+z_n)$ and symmetrizing then yields the desired result. □

We also need to compute the commutators of various pairs of the operators δ,ϖ_k,V_k,W_k, and $\overline{W}_k$.

Lemma 4.2.3. *The commutators $[\delta,V_k]$, $[\delta,\varpi_l]$, and $[V_{-1},\varpi_l]$ all vanish (for all $k\in\mathbb{Z}$ and odd $l\ge 1$). Also, for any $I\in\mathcal{A}$, the invariants $\langle[\delta,W_k]I\rangle^\bullet$ and $\langle[\delta,\overline{W}_k]I\rangle^\bullet$ vanish.*

Proof. The first part of this lemma is a completely straightforward computation using the supercommutativity relations (3.2). We perform the first calculation as an example:

$$
\begin{aligned}
[\delta, V_k] &= \left[\sum_{i\in\mathbb{Z}}(-1)^i\tau_i(1)\tau_{-i}(1), \sum_{j\in\mathbb{Z}}(-1)^j\binom{j}{k+1}(\tau_j(1)\tau_{k-1-j}(\omega)-\tau_j(\alpha)\tau_{k-1-j}(\beta))\right]\\
&= 2\sum_{j\in\mathbb{Z}}(-1)^{j-k}(-1)^j\binom{j}{k+1}(-1)^{k-1-j}\tau_{k-j}(1)\tau_j(1)\\
&= 2\sum_{j\in\mathbb{Z}}(-1)^{j+1}\binom{j}{k+1}\tau_{k-j}(1)\tau_j(1)\\
&= \sum_{j\in\mathbb{Z}}(-1)^{j+1}\binom{j}{k+1}\tau_{k-j}(1)\tau_j(1)+\sum_{j\in\mathbb{Z}}(-1)^{k-j+1}\binom{k-j}{k+1}\tau_j(1)\tau_{k-j}(1)\\
&= 0
\end{aligned}
$$

because $\binom{k-j}{k+1} = (-1)^{k+1}\binom{j}{k+1}$.

The second part of this lemma is slightly more complicated, since the two commutator operators are do not themselves vanish. We will only go through the case involving W_k, since the other one is identical. An analogous computation to the one performed above yields the identity

$$
[\delta, W_k] = 2\sum_{j\in\mathbb{Z}}(-1)^{j+1}\binom{j}{k+1}\tau_j(\beta)\tau_{k-j}(1).
$$

Switching back to derivative notation, we see that we need to prove the relation

$$
\left\langle \sum_{j\geq 0}\binom{j+k+1}{k+1}\left(\tau_{j+k+1}(\beta)\frac{d}{d\tau_j(\omega)}+\tau_{j+k+1}(1)\frac{d}{d\tau_j(\alpha)}\right) I\right\rangle^{\bullet} = 0 \tag{4.7}
$$

for any descendent insertion I.

In order to do this, we apply Proposition 3.3.2 to the resulting descendent invariant and for each partition of the descendents in the resulting sum, we consider the part containing the descendent coming from the operator $[\delta, W_k]$ of (4.7). This is a subset of the descendent factors which (replacing the descendent coming from the operator with the descendent removed by the operator) originally either contained one descendent of α, one descendent of ω, and some number of descendents of 1, or contained two descendents of α, one descendent of β, and some number of descendents of 1. In either case, there are exactly two different ways for the operator to act on this subset of the descendent factors, and it is easily checked that their contributions cancel, yielding the desired vanishing. □

We can now give a useful result that essentially states that δ ignores Hodge insertions.

Proposition 4.2.4. *Let $\Lambda \in \mathrm{End}(\mathcal{A})$ be any polynomial in $\varpi_1, \varpi_3, \ldots$, let $I \in \mathcal{A}$ be a descendent insertion, and let $g \geq 0$. Then $\langle \delta \Lambda I \hbar^{g-1} \rangle^\bullet = \langle \Lambda I' \hbar^{g-1} \rangle^\bullet$ for*

$$I' = \left(\hbar^{-1} \left(\sum_{\substack{k \geq 0 \\ \gamma \in \mathcal{B}}} \tau_{k-1}(\gamma) \frac{d}{d\tau_k(\gamma)} \right)^2 + \frac{d}{d\tau_0(\omega)} - 2 \sum_{k \geq 0} \tau_{k+1}(1) \frac{d}{d\tau_k(\omega)} \right) I.$$

Proof. First, observe that

$$I' = (\delta + \hbar^{-1}((\tau_0(1) + V_{-1})^2 - [\tau_0(1), V_{-1}] - \tau_0(1)^2))I.$$

Thus

$$\langle \Lambda I' \hbar^{g-1} \rangle^\bullet = \langle \Lambda(\delta + \hbar^{-1}(2V_{-1}\tau_0(1) + V_{-1}^2)) I \hbar^{g-1} \rangle^\bullet.$$

The result then follows from using the commutativity of ϖ_k with V_{-1} and δ (see Lemma 4.2.3) along with the Virasoro constraint for V_{-1} (Theorem 3.3.1). □

We are now ready to prove our main theorem.

Proof of Theorem 4.2.1. We first assume that I is stationary, so let $I = \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \in \mathcal{S}$. The reduction of δI has two parts: those terms coming from the $\tau_0(1)\tau_0(1)$ summand of δ and those coming from the other summands. One can check (using Proposition 4.2.4, for instance) that

$$\begin{aligned}(\delta I)^{\mathrm{red}} &= \sum_{i=1}^{n} \tau_{k_1}(\omega) \cdots \tau_{k_i - 2}(\omega) \cdots \tau_{k_n}(\omega) + 2 \sum_{1 \leq i < j \leq n} \tau_{k_1}(\omega) \cdots \tau_{k_i - 1}(\omega) \cdots \tau_{k_j - 1}(\omega) \cdots \tau_{k_n}(\omega) \\ &\quad - 2 \sum_{1 \leq i \neq j \leq n} \binom{k_i + k_j + 1}{k_i} \tau_{k_i + k_j}(\omega) \prod_{l \neq i,j} \tau_{k_l}(\omega).\end{aligned}$$

The theorem statement then follows directly from taking the coefficient of $z_1^{k_1+1} \cdots z_n^{k_n+1}$ in Lemma 4.2.2.

We now prove the general case. First, note that the commutativity of ϖ_k with δ (for odd k), as proven in Lemma 4.2.3 and used in Proposition 4.2.4, allows us to replace the Hodge classes by descendent factors using Corollary 3.4.2, so without loss of generality we can assume that $I \in \mathcal{A}$ is a descendent insertion.

Now, any descendent insertion can be written in the form

$$I = S + \sum_{k \geq -1} (V_k A_k + W_k B_k + \overline{W}_k \overline{B}_k), \tag{4.8}$$

where $S \in \mathcal{S}$ is stationary, $A_k, B_k, \overline{B}_k \in \mathcal{A}$, and all but finitely many terms are zero. We have already verified the theorem for S, and the commutativity statements in Lemma 4.2.3 combined with the Virasoro constraints Theorem 3.3.1 imply that

$$\langle \delta V_k A_k \rangle^\bullet = \langle V_k \delta A_k \rangle^\bullet = 0 = \partial \langle V_k A_k \rangle^\bullet$$

(and similarly for W_k, $\overline{W}_k$), so the theorem also holds for the other terms in the decomposition (4.8). The theorem then holds for I by linearity. □

4.3 Asymptotics

As a consequence of the simple bracket relation (4.5), ∂ controls the asymptotics of the coefficients of a quasimodular form. To see this, suppose that $f \in \mathrm{QM}_k$ is a weight k quasimodular form, and note that f can be (uniquely) written in the form

$$f = f_0 + \theta f_2 + \cdots + \theta^{\frac{k}{2}-1} f_{k-2}, \tag{4.9}$$

where each f_m is a modular form of weight $k-m$ for $0 \leq m \leq k-4$ and f_{k-2} is a weight 2 quasimodular form (and thus a scalar multiple of E_2).

Let a_m be the constant term of f_m, and let

$$f^{\approx} = a_0 E_k + a_2 \theta E_{k-2} + \cdots + a_{k-2} \theta^{\frac{k}{2}-1} E_2.$$

Observe that $f - f^{\approx}$ is a linear combination of derivatives of cusp forms, so the Ramanujan bounds (proven by Deligne in [4]) give that the dth Fourier coefficient of $f - f^{\approx}$ is $\mathbf{O}(d^{\frac{k-1}{2}+\epsilon})$ for any $\epsilon > 0$. In combination with the exact formula for the Fourier expansion of $f^{\approx}$, we have that

$$[q^d]f = \sum_{i=0}^{\frac{k}{2}-1} a_{2i} \left(-\frac{2k-4i}{B_{k-2i}} \right) d^i \sigma_{k-2i-1}(d) + \mathbf{O}(d^{\frac{k-1}{2}+\epsilon}). \tag{4.10}$$

We now return to the derivation ∂. The coefficients of the asymptotic expansion (4.10) can be obtained by taking the constant term of $\partial^i f$ for each i:

Lemma 4.3.1. *Let $f \in \mathrm{QM}_k$ be a weight k modular form and let $a_0, a_2, \ldots, a_{k-2}$ be defined as above. Then for $0 \leq i \leq \frac{k}{2} - 1$,*

$$[q^0]\partial^i f = (-2)^i (i!)(k-2i)(k-2i+1)\cdots(k-i-1) a_{2i}.$$

Proof. Suppose that g is a modular form of some weight l; if $l = 2$, then we permit g to be only quasimodular. Then for any $i \geq 1$, we can apply (4.5) to compute that

$$\partial \theta^j g = -2((l+2j-2) + (l+2j-4) + \cdots + l)\theta^{j-1} g,$$

since $\theta \partial g = 0$, even when $l = 2$. We can then repeatedly apply this result to (4.9) to obtain a similar decomposition of $\partial^i f$ into derivatives of modular forms. Only the first term will have a nonzero constant term, and multiplying together the appropriate factors yields the desired result. □

If f was obtained as an elliptic curve invariant, then $\partial^i f$ will also have a natural interpretation as an invariant by Theorem 4.2.1, and the constant term will just be the degree 0 invariant, which is often easier to compute. We state the resulting asymptotics:

Proposition 4.3.2. *Suppose that $I \in \mathcal{A}$ is an insertion such that $\langle I \rangle^\bullet$ is a quasimodular form of weight k. Then the individual invariants $\langle I \rangle^\bullet_{d\omega}$ have asymptotic expansion*

$$\langle I \rangle^\bullet_{d\omega} = \sum_{i=0}^{\frac{k}{2}-1} \frac{(k-2i)!}{(-2)^{i-1} i! (k-i-1)! B_{k-2i}} \langle \delta^i I \rangle^\bullet_0 d^i \sigma_{k-2i-1}(d) + \mathbf{O}(d^{\frac{k-1}{2}+\epsilon})$$

(for all $\epsilon > 0$).

As we will see in the next section, this method is a highly effective way to produce complete asymptotic expansions of the form (4.10) for elliptic curve invariants. One of the reasons why Proposition 4.3.2 is so nice is the following general result on the so-called "degree 0" invariants. This proposition was repeatedly used in [10] in the special case of the trivial product $X = * \times X$, but the general argument is identical.

Proposition 4.3.3. *Let $X = X_1 \times X_2$ be a product of smooth projective varieties. Let $\beta \in H_2(X)$ be the image of a class $\beta \in H_2(X_1)$ under the fiber map $X_1 \to X$. Then $\overline{M}_{g,n}(X,\beta) = X_2 \times \overline{M}_{g,n}(X_1,\beta)$, and the corresponding virtual fundamental classes are related by*

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} = e(T_{X_2} \otimes \mathbb{E}^\vee) \cap ([X_2] \times [\overline{M}_{g,n}(X_1,\beta)]^{\mathrm{vir}}),$$

where $e(T_{X_2} \otimes \mathbb{E}^\vee)$ is the Euler class of the vector bundle $T_{X_2} \otimes \mathbb{E}^\vee$.

In the case $X_1 = *$, $X_2 = E$, we obtain the following corollary about connected degree 0 invariants on an elliptic curve:

Corollary 4.3.4. *Let $\Lambda \in \mathcal{H}$ and $I = \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)$ be insertions of Hodge and descendent type on E. If $\gamma_1 \cup \cdots \cup \gamma_n = \epsilon\omega$ for $\epsilon = \pm 1$, then*

$$\langle \Lambda I \rangle^E_{g,0} = \int_{\overline{M}_{g,n}} \epsilon(-1)^g \lambda_g \Lambda \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

Otherwise, $\langle \Lambda I \rangle^E_{g,0} = 0$.

This allows us to compute the asymptotic expansion of an arbitrary elliptic curve invariant in terms of integrals on the moduli space of curves.

Our approach to the asymptotics of the elliptic curve invariants should be compared with the work of Eskin and Okounkov ([5]) on the asymptotics of Hurwitz numbers, which are closely related.

4.4 Applications and conjectures

Although Proposition 4.3.2 is generally applicable, the resulting asymptotic expansions are particularly nice in certain special cases.

We will find the folllowing fundamental geometric result to be extremely useful.

Lemma 4.4.1. *The genus g elliptic invariant $\langle \lambda_g I \hbar^{g-1} \rangle_{d\omega}$ vanishes for all insertions $I \in \mathcal{A}'$ and all degrees $d \geq 0$.*

Proof. Let $X = E \times E$. By Proposition 4.3.3, we have

$$[\overline{M}_{g,n}(X,(d\omega,0))]^{\mathrm{vir}} = (-1)^g \lambda_g \cap [\overline{M}_{g,n}(E,d\omega)]^{\mathrm{vir}} \times [E].$$

As a consequence, we have the equality

$$\langle \lambda_g I \hbar^{g-1} \rangle^E_{g,d\omega} = \langle (-1)^g I' \hbar^{g-1} \rangle^X_{g,(d\omega,0)},$$

where I' has been formed by taking I and multiplying one of the classes pulled back from X (and to there from E via the projection onto the first component) by the fundamental class of the second elliptic factor of X. However, all Gromov-Witten invariants of an abelian surface vanish. This is for the same reason that all (nonreduced) Gromov-Witten invariants of a K3 surface vanish, as any abelian surface can be symplectically deformed so that it no longer contains curves in a given class. $\square$

Genus g invariants containing a λ_{g-1} insertion thus can be thought of as being on the verge of vanishing by the above lemma. One consequence of this is that these invariants have extremely short asymptotic expansions of the type produced by Proposition 4.3.2, since multiplying by δ tends to decrease the genus of an insertion. We start with the simplest possible example:

Theorem 4.4.2. *The elliptic invariant $\langle \lambda_{g-1}\tau_{g-1}(\omega)\rangle$ is modular for any $g \geq 1$. Moreover, the individual coefficients satisfy*

$$\langle \lambda_{g-1}\tau_{g-1}(\omega)\rangle_{d\omega} = \frac{g!}{(2g)!2^{g-2}}\sigma_{2g-1}(d) + \mathbf{O}(d^{g-\frac{1}{2}+\epsilon})$$

for all $\epsilon > 0$.

Proof. Applying Theorem 4.2.1, we have that $\partial\langle \lambda_{g-1}\tau_{g-1}(\omega)\rangle = \langle \lambda_{g-1}\tau_{g-3}(\omega)\rangle$ is a genus $g-1$ invariant and thus vanishes by Lemma 4.4.1. Thus $\langle \lambda_{g-1}\tau_{g-1}(\omega)\rangle$ is modular. The second half of the theorem follows from Proposition 4.3.2 and the evaluation of the degree 0 invariant $\langle \lambda_{g-1}\tau_{g-1}(\omega)\hbar^{g-1}\rangle_0^E$. By Corollary 4.3.4, this invariant is equal to the Hodge integral

$$\int_{\overline{M}_{g,1}} (-1)^g \lambda_g \lambda_{g-1}\psi^{g-1}, \tag{4.11}$$

which was calculated by Faber ([8]) to be $-\frac{(g-1)!B_{2g}}{2^g(2g)!}$. Multiplying all the constant factors together gives the claimed result. $\square$

The strength of these asymptotics may suggest the following conjecture:

Conjecture 4.4.3. *For any $g \geq 1$,*

$$\langle \lambda_{g-1}\tau_{g-1}(\omega)\rangle = \frac{g!}{2^{g-1}}C_{2g}.$$

In other words, the asymptotic expansion provided by Proposition 4.3.2 may actually be exact in this case. We will later see further motivation for believing this conjecture, which was checked for $g \leq 8$ using a Maple program ([3]) written by Bryan and Pandharipande for computing Gromov-Witten invariants of curves.

In general, any genus g invariant containing a λ_{g-1} insertion will demonstrate similar behavior.

Theorem 4.4.4. *Let $I \in \mathcal{A}'$ be any monomial in the Hodge classes and the elliptic descendent invariants and suppose that $g \geq 1$. Then there exists $C \in \mathbb{Q}$ and $e \geq 0$ such that*

$$\langle \lambda_{g-1} I \rangle^E_{g,d\omega} = C d^e \sigma_{2g-1}(d) + \mathbf{O}(d^{e+g-\frac{1}{2}+\epsilon})$$

for any $\epsilon > 0$. If $I = \tau_{k_1}(\omega) \cdots \tau_{k_m}(\omega) \tau_{k_{m+1}+1}(1) \cdots \tau_{k_n+1}(1)$ with $k_1, \ldots, k_m, k_{m+1}+1, \ldots, k_n+1 \geq 1$ and $k_1 + \ldots + k_n = g-1$, then $e = m-1$ and

$$C = \frac{(2g+n-3)! \sum_{i=1}^m (2k_i+1)}{2^{2g-2}(2g+m-2)! \prod_{i=1}^n (2k_i+1)!!},$$

where $(2k_i+1)!! = 1 \cdot 3 \cdots (2k_i+1)$.

Proof. Let $I = \Lambda I'$, where $\Lambda \in \mathcal{H}$ is an element of the Hodge algebra and $I' \in \mathcal{A}$ is a product of descendent classes. Applying Proposition 4.2.4 helps us to expand $\langle \delta^j \lambda_{g-1} \Lambda I' \rangle^E_g$. Most of the resulting terms vanish because the genus decreases to $g-1$; the only terms of the operator described in Proposition 4.2.4 that do not decrease the genus are

$$\eta := \frac{d}{d\tau_0(\omega)} - 2 \sum_{k \geq 0} \tau_{k+1}(1) \frac{d}{d\tau_k(\omega)}.$$

Although Proposition 4.2.4 only applies directly to disconnected invariants, we can use Proposition 3.1.1 to see that we still have $\langle \delta^j \lambda_{g-1} I \rangle^E_g = \langle \lambda_{g-1} \Lambda \eta^j I' \rangle^E_g$. We then apply Corollary 4.3.4 to see that the degree 0 term of this invariant vanishes unless the product of the evaluation classes left in some term of $\eta^j I'$ is equal to ω, which clearly happens for at most one value of j. Then Proposition 4.3.2 implies that the asymptotic expansion of $\langle \lambda_{g-1} I \rangle^E_{g,d\omega}$ has at most one term in it, as desired.

For the second part of this theorem, this value of j is clearly equal to $m-1$, and Corollary 4.3.4 implies that

$$\langle \lambda_{g-1} \eta^{m-1} I \rangle^E_{g,0} = (-2)^{m-1}(m-1)! \sum_{i=1}^m \int_{\overline{M}_{g,n}} (-1)^g \lambda_g \lambda_{g-1} \psi_1^{k_1+1} \cdots \psi_i^{k_i} \cdots \psi_n^{k_n+1}. \tag{4.12}$$

These integrals on the moduli space of curves were computed by Getzler and Pandharipande in [10] assuming the Virasoro conjecture for $\mathbb{P}^2$ in degree 0, which has since been proven. Their idea was that Proposition 4.3.3 can be used to relate integrals of this form to the descendent invariants of $\mathbb{P}^2$, which could then be computed using the Virasoro constraints. In this way, one arrives at the formula

$$\int_{\overline{M}_{g,n}} (-1)^g \lambda_g \lambda_{g-1} \psi_1^{l_1} \cdots \psi_n^{l_n} = \frac{(2g+n-3)!(2g-1)!!}{(2g-1)!(2l_1-1)!! \cdots (2l_n-1)!!} \int_{\overline{M}_{g,1}} (-1)^g \lambda_g \lambda_{g-1} \psi^{g-1} \tag{4.13}$$

(for positive l_i) expressing the given integrals in terms of an integral which we have already encountered in (4.11). The combination of (4.12), (4.13), and Proposition 4.3.2 then gives the claimed value for C. $\square$

The corresponding general conjecture can be stated more succinctly:

Conjecture 4.4.5. *For any insertion $I \in \mathcal{A}$, the asymptotic expansion provided by Proposition 4.3.2 for $\langle \lambda_g I \rangle^E_{g,d\omega}$ is exact.*

In general, when is the asymptotic expansion of Proposition 4.3.2 exact? In other words, which elliptic invariants $\langle I \rangle$ are linear combinations of derivatives of Eisenstein series? We have just conjectured that this is the case whenever I is a genus g invariant containing a λ_{g-1} factor, but we can only prove the following special case of this conjecture.

Proposition 4.4.6. $\langle \lambda_{g-1}\lambda_{g-2}\tau_1(\omega)\rangle^E = |B_{2g-2}|\binom{2g}{2}C_{2g}$

Proof. As observed by Faber (see the proof of Lemma 1 in [8]), the Mumford relations for the Hodge classes imply that

$$\sum_{k\geq 0}(2k+1)!\,\mathrm{ch}_{2k+1}\,t^{2k} = \left(\sum_{i=0}^{g} i\lambda_i t^{i-1}\right)\left(\sum_{i=0}^{g}\lambda_i(-t)^i\right).$$

Taking the coefficient of t^{2g-4} yields that

$$(2g-3)!\,\mathrm{ch}_{2g-3} = ((-1)^{g-3}g+(-1)^g(g-3))\lambda_g\lambda_{g-3}+((-1)^{g-2}(g-1)+(-1)^{g-1}(g-2))\lambda_{g-1}\lambda_{g-2},$$

so

$$\lambda_{g-1}\lambda_{g-2} = (-1)^g(2g-3)!\,\mathrm{ch}_{2g-3} + 3\lambda_g\lambda_{g-3}.$$

The invariant $\langle \lambda_g\lambda_{g-3}\tau_1(\omega)\rangle^E$ vanishes by Lemma 4.4.1, so Proposition 3.5.1 gives

$$\langle \lambda_{g-1}\lambda_{g-2}\tau_1(\omega)\rangle^E = (-1)^g(2g-3)!\langle \mathrm{ch}_{2g-3}\,\tau_1(\omega)\rangle^E = |B_{2g-2}|\binom{2g}{2}C_{2g},$$

as desired. □

All of these invariants containing λ_{g-1} can be interpreted using Proposition 4.3.3 as invariants on $E\times\mathbb{P}^1$ representing the class $(d\omega, 0)$ in much the same way as the λ_g invariants were interpreted as invariants on $E\times E$ in the proof of Lemma 4.4.1. More precisely,

$$\langle \lambda_{g-1} I \hbar^{g-1}\rangle^E_{d\omega} = (-1)^{g-1}\frac{1}{2}\langle I\hbar^{g-1}\rangle^{E\times\mathbb{P}^1}_{(d\omega,0)}. \tag{4.14}$$

The Virasoro conjectures have not yet been proven for $E\times\mathbb{P}^1$ (even in this case), but we can use (4.14) to prove the following consequence of the Virasoro conjectures:

Proposition 4.4.7. *The Virasoro conjectures for the pair $(E\times\mathbb{P}^1,(d\omega,0))$ for $d\geq 0$ imply Conjecture 4.4.3.*

Proof. We have already computed in Proposition 4.4.6 that the elliptic invariant

$$\langle \lambda_{g-1}\lambda_{g-2}\tau_1(\omega)\hbar^{g-1}\rangle = \langle \lambda_{g-1}\lambda_{g-2}\tau_1(\omega)\hbar^{g-1}\rangle^{\bullet}$$

is a multiple of the Eisenstein series C_{2g}. We now compute this elliptic invariant in a different way. The Hodge class λ_{g-2} can be written as a polynomial in the Chern characters ch_k, which we then replace by descendent factors using Proposition 3.4.1. By Lemma 4.4.1, all the terms in which the genus decreases vanish. This has the consequence that the resulting invariants after removing all of the ch_k factors are all of the form

$$\langle \lambda_{g-1}\tau_k(\omega)\tau_{l_1}(1)\cdots\tau_{l_n}(1)\hbar^{g-1}\rangle.$$

The observations before this proposition together with the Virasoro constraints for $E \times \mathbb{P}^1$ then allow us to remove the $\tau_{l_i}(1)$ insertions, and we conclude that $\langle \lambda_{g-1}\lambda_{g-2}\tau_1(\omega)\hbar^{g-1}\rangle$ is some multiple of $\langle \lambda_{g-1}\tau_{g-1}(\omega)\hbar^{g-1}\rangle$, and thus $\langle \lambda_{g-1}\tau_{g-1}(\omega)\hbar^{g-1}\rangle$ is a multiple of the Eisenstein series C_{2g}, as desired. $\square$

We also have the following unexpected example of an exact asymptotic expansion.

Proposition 4.4.8. *Let $d, g \geq 1$. Then*

$$\begin{aligned}\langle \tau_{2g-2}(\omega)\rangle^{E}_{g,d\omega} &= \frac{2^{2g-2}}{(2g-1)!}\sum_{mn=d}(2m-n)^{2g-1}\\ &= \frac{2^{2g-2}}{(2g-1)!}\sum_{i=0}^{g-1}(-1)^i\binom{2g-1}{i}(2^{2g-1-i}-2^i)d^i\sigma_{2g-1-2i}(d).\end{aligned}$$

Proof. Recall that

$$\Theta(z) = (e^{\frac{z}{2}} - e^{-\frac{z}{2}})\prod_{k\geq 1}\frac{(1-e^zq^k)(1-e^{-z}q^k)}{(1-q^k)^2}.$$

The formula to be proven is then equivalent to the easily proven identity

$$\prod_{k\geq 1}\frac{(1-q^k)^2}{(1-xq^k)(1-x^{-1}q^k)} = 1 + (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\sum_{m,n\geq 1}(x^{m-\frac{n}{2}} - x^{\frac{n}{2}-m})q^{mn}$$

upon setting $x = e^z$. $\square$

Chapter 5

K3 surfaces

5.1 Reduced invariants and the KKV conjecture

We now consider the case of Gromov-Witten invariants on a K3 surface X in greater depth. Recall that these invariants must be defined by intersecting with a "reduced" virtual class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{red}}$ of dimension $g+n$.

By analogy with our special generatingfunctionological notation for elliptic curve invariants, we let

$$\langle I\rangle_g^{K3} = \sum_{h\geq 0} \langle I\rangle_{g,\beta}^{K3} q^{h-1},$$

where β is chosen to be a primitive class with $\beta^2 = 2h-2$ and g is often omitted and then is determined by the dimension of the insertion I.

Thus the Bryan-Leung example of Section 2.1 can be rewritten as

$$\langle \tau_0(p)\tau_0(p)\cdots\tau_0(p)\rangle_n^{K3} = \frac{(\theta C_2)^n}{\Delta}, \tag{5.1}$$

where $\Delta = q\prod_{k\geq 1}(1-q^k)^{24}$ is Ramanujan's delta-function, the unique weight 12 cusp form with Fourier expansion $q + \mathbf{O}(q^2)$.

The KKV conjecture is the most important open question in the Gromov-Witten theory of the K3 surface. It is equivalent to the following evaluation of the pure Hodge invariants $\langle\lambda_g\rangle^{K3}$:

$$\sum_{g\geq 0} \langle\lambda_g\rangle^{K3} z^{2g-2} = \frac{1}{\Delta\cdot\Theta(z)^2}.$$

See Conjecture 2 of [14] for an alternative description in terms of BPS counts.

Note that the $g=0$ piece of the KKV conjecture coincides with the $n=0$ case of the theorem of Bryan and Leung; this was first conjectured by Yau and Zaslow in [20].

It is not a coincidence that in both of the above examples, all of the K3 invariants are of the form $\frac{f}{\Delta}$ for f a quasimodular form. A connection between the elliptic curve invariants of the preceding chapters and these K3 invariants, first used by Bryan and Leung

([2]) to prove (5.1), was recently expanded by Maulik and Pandharipande ([13]) to give an algorithm for writing $\Delta\langle I\rangle^{K3}$ in terms of elliptic curve invariants for any insertion I. This algorithm unfortunately does not seem to be directly applicable to the Hodge integrals of the KKV conjecture, but we will see in the next section that it is very useful for evaluating the stationary K3 invariants.

5.2 The elliptic connection

Bryan and Leung proved their result by using a K3 surface with an elliptic fibration with section; in other words, they chose (via deformation arguments) to use a K3 surface X equipped with a map $X \to \mathbb{P}^1$ such that all but 24 fibers are smooth elliptic curves and the 24 singular fibers are nodal rational curves, and such that there is a section $\mathbb{P}^1 \to X$.

Let $F, S \in H_2(X)$ be the classes of the fibers and the section respectively. Then $\beta = hF+S$ is a primitive element of the Picard lattice satisfying $\beta^2 = 2h(F\cdot S)+S^2 = 2h-2$, so it can be used for computing the Gromov-Witten invariants of X. This is particularly nice because the moduli space of stable maps $\overline{M}_{g,n}(X,\beta)$ is relatively simple. Stable maps $f : C \to X$ representing β must consist of one irreducible component of genus zero mapping isomorphically along the section and some number of other components mapped onto individual fibers, with the sum of the degrees of these other maps being h. Moreover, since only curves of positive genus can map onto an elliptic curve, all but at most g of these other irreducible components must map to nodal fibers.

The invariants considered by Bryan and Leung can be interpreted as counting the number of genus n maps whose images contain n given generic points on X. If these n points are chosen to belong to distinct smooth fibers of an elliptic fibration of the type described above, then all such maps must consist of isogenies from elliptic curves to each of the n elliptic fibers, connected by the rational section curve and possibly containing additional rational components mapping to the singular fibers.

Thus these invariants can be rewritten as a sum as follows:

$$\langle\tau_0(p)\cdots\tau_0(p)\rangle^{K3}_{n,hF+S} = \sum_{d_1+\cdots+d_n+e_1+\cdots+e_{24}=h} A_{d_1}\cdots A_{d_n}P_{e_1}\cdots P_{e_{24}}, \tag{5.2}$$

where A_d is the number of degree d isogenies from some elliptic curve with two marked points (the intersection with the section curve and the actual marked point) to a given elliptic curve with two marked points) and P_e is a similar (but more complicated) count of the contribution (with multiplicity) from the degree e maps to a singular fiber.

Now, it is easily computed that $A_d = d\sigma_1(d)$, since there are $\sigma_1(d)$ index d sublattices of a given 2-dimensional lattice and then there are d choices for the marked points on the source curve. Alternatively, note that $A_d = [q^d]\langle\tau_0(\omega)\tau_0(\omega)\rangle = [q^d]\theta C_2$. Through a more involved computation, Bryan and Leung determined that P_e is simply the partition number

$p(e) = [q^e]\prod_{k\geq 1}(1-q^k)^{-1}$. As a consequence, they obtained from (5.2) that

$$\langle \tau_0(p)\cdots\tau_0(p)\rangle_n^{K3} = q^{-1}(\theta C_2)^n\left(\prod_{k\geq 1}(1-q^k)^{-1}\right)^{24} = \frac{(\theta C_2)^n}{\Delta}.$$

In a forthcoming paper, Maulik and Pandharipande ([13]) describe how to extend this method of Bryan and Leung to reduce any K3 invariant to a product of elliptic curve invariants and a $\frac{1}{\Delta}$ factor from the singular fibers, as above. This method works particularly well for the stationary invariants, when it describes quasimodular forms T_k of weight $2k+4$ such that

$$\langle \tau_{k_1}(p)\cdots\tau_{k_n}(p)\rangle^{K3} = \frac{1}{\Delta}T_{k_1}\cdots T_{k_n} \tag{5.3}$$

for any $k_1,\ldots,k_n \geq 0$.

Here T_k can be obtained in terms of elliptic curve invariants by

$$T_k = \sum_{\substack{i,j\geq 0\\ 2i+j\leq k}} (-1)^{i+j}\frac{C_2^i}{i!}\langle \lambda_j\tau_k(\omega)\tau_{k-2i-j}(\omega)\rangle^E.$$

Straightforward computation using the methods of the previous chapters gives the first few of these quasimodular factors:

$$\begin{aligned}
T_0 &= \langle \tau_0(\omega)\tau_0(\omega)\rangle^E = \theta C_2 = -2C_2^2 + 10C_4\\
T_1 &= \langle \tau_1(\omega)\tau_1(\omega)\rangle^E - \langle \lambda_1\tau_1(\omega)\tau_0(\omega)\rangle^E = \theta\left(\frac{2}{3}C_2^2 - \frac{1}{3}C_4\right) = -\frac{8}{3}C_2^3 + 16C_2C_4 - 7C_6\\
T_2 &= \langle \tau_2(\omega)\tau_2(\omega)\rangle^E - \langle \lambda_1\tau_2(\omega)\tau_1(\omega)\rangle^E + \langle \lambda_2\tau_2(\omega)\tau_0(\omega)\rangle^E - C_2\langle \tau_2(\omega)\tau_0(\omega)\rangle^E\\
&= \theta\left(-\frac{1}{6}C_2^3 - \frac{11}{5}C_2C_4 - \frac{11}{10}C_6\right) = C_2^4 + 17C_2^2C_4 - 33C_2C_6 - \frac{330}{7}C_4^2.
\end{aligned}$$

In general, it seems difficult to obtain an exact formula for T_k.

5.3 Computations

In this section, we perform the computations necessary to verify the KKV conjecture for $g \leq 3$. The basic idea will be to write the Hodge class λ_g as a linear combination of boundary strata. This will allow us to express the Hodge integrals in terms of relatively simple descendent invariants on the K3 surface, which can then be evaluated through a combination of methods, including use of the theorem of Bryan and Leung (5.1) and the generalization (5.3). We will also need the formulas (4.4) for the action of the differentiation operator θ on the first few Eisenstein series, together with the identity

$$\theta\left(\frac{1}{\Delta}\right) = \frac{24C_2}{\Delta}.$$

It will be convenient to fix a basis for the cohomology of our $K3$ surface X. Let $\gamma_0 = 1 \in H^0(X)$ and let $\gamma_{23} = [p] \in H^4(X)$ be the fundamental class of X. Let $\gamma_1, \cdots, \gamma_{22}$ be any basis for $H^2(X)$. Then we can define the dual basis $\gamma_i^\vee \in H^*(X)$ for $0 \leq i \leq 23$ by $\gamma_i \gamma_j^\vee = \delta_{ij}[p]$. Of course, $\gamma_0^\vee = \gamma_{23}$ and $\gamma_{23}^\vee = \gamma_0$.

We begin by noting that when $g = 0$, the KKV conjecture is simply the Yau-Zaslow conjecture $\langle 1 \rangle^{K3} = \frac{1}{\Delta}$, proven by Bryan and Leung (see [20], [2]).

The case $g = 1$ is significantly more involved. We want to rewrite λ_1 in terms of Q-classes of boundary strata on a moduli stack of curves. However, $\overline{M}_1$ is not stable, so we add a marked point: by the divisor equation,

$$\int_{[\overline{M}_{1,0}(X,\beta)]^{\mathrm{red}}} \lambda_1 = \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{red}}} \lambda_1 \operatorname{ev}_1^*(\beta^\vee),$$

where $\beta^\vee \cdot \beta = 1$.

Now, let $\delta_0 \in H^2(\overline{M}_{1,1})$ denote the Q-class $[\Delta_0]$, where Δ_0 is the boundary locus of genus 0 curves with one node (and one marked point), which is just a single point. It is well known that $\lambda_1 = \frac{1}{12}\delta_0$, so we can remove the λ_1 insertion, restrict to maps from Δ_0, and resolve the node to obtain

$$\int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{red}}} \lambda_1 \operatorname{ev}_1^*(\beta^\vee) = \frac{1}{12} \cdot \frac{1}{2} \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{red}}} \operatorname{ev}_1^*(\beta^\vee)(\operatorname{ev}_2 \times \operatorname{ev}_3)^*([D]),$$

where $[D] = \sum_{i=0}^{23} (\gamma_i, \gamma_i^\vee) \in H^4(X \times X)$ is the Poincaré dual of the diagonal $X \subset X \times X$. (Also, the extra factor of $\frac{1}{2}$ is because there are two different ways of labeling the two new marked points.) Now, the genus zero invariants involving pullbacks of $\gamma_{23} = [p]$ all vanish because there are only a finite number (determined by Yau-Zaslow) of rational curves on X representing β, so such curves cannot be constrained to pass through a generic point of X. This means that we only have the terms

$$\frac{1}{24} \sum_{i=1}^{22} \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{red}}} \operatorname{ev}_1^*(\beta^\vee) \operatorname{ev}_2^*(\gamma_i) \operatorname{ev}_3^*(\gamma_i^\vee).$$

Applying the divisor equation again gives that this is just

$$\frac{1}{24} \sum_{i=1}^{22} (\beta \cdot \gamma_i)(\beta \cdot \gamma_i^\vee) \langle 1 \rangle^{K3},$$

which simplifies further upon using Yau-Zaslow to

$$\begin{aligned} \frac{1}{24}(\beta \cdot \beta)[q^{h-1}]\frac{1}{\Delta} &= \frac{h-1}{12}[q^{h-1}]\frac{1}{\Delta} \\ &= [q^{h-1}]\frac{1}{12}\theta\left(\frac{1}{\Delta}\right) \\ &= [q^{h-1}]\frac{2C_2}{\Delta}, \end{aligned}$$

as predicted by the KKV conjecture.

In genus 2 we do not have to use the same trick of introducing a marked point, so we just want to write λ_2 in terms of boundary classes on $\overline{M}_2$. Here the relevant boundary strata are Δ_{00}, the generic element of which is a genus 0 curve with 2 nodes, and Δ_{01}, where the generic element is a genus 0 curve with 1 node intersecting a smooth genus 1 curve in a single point. The corresponding Q-classes are $\delta_{00}, \delta_{01} \in H^4(\overline{M}_2)$. It turns out that

$$\lambda_2 = \frac{1}{120}(\delta_{00} + \delta_{01})$$

(see section 8 of [15] for more details).

Again, we can replace λ_2 by the classes δ_{00} and δ_{01} and then remove these classes by restricting to maps from curves in the corresponding boundary loci; after resolving the singularities of the source curves, we see that

$$\begin{aligned} \langle \lambda_2 \rangle_\beta^{K3} =& \frac{1}{120} \cdot \frac{1}{8} \int_{[\overline{M}_{0,4}(X,\beta)]^{\mathrm{red}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*([D])(\mathrm{ev}_3 \times \mathrm{ev}_4)^*([D]) \\ &+ \frac{1}{120} \cdot \frac{1}{2} \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{red}} \times [\overline{M}_{0,3}(X,0)]^{\mathrm{vir}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*([D])(\mathrm{ev}_3 \times \mathrm{ev}_4)^*([D]). \end{aligned} \tag{5.4}$$

Remark. In the second term, it should be observed that the curve class β cannot split nontrivially between the two irreducible components because it is primitive, and thus one of the two components must be contracted to a point; this can only be the rational component because its moduli space $\overline{M}_{0,3}$ has dimension 0.

We now compute the two terms of (5.4). The first term is completely analogous to the calculation in genus 1, and we obtain

$$\begin{aligned} \frac{1}{120} \cdot \frac{1}{8} \int_{[\overline{M}_{0,4}(X,\beta)]^{\mathrm{red}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*([D])(\mathrm{ev}_3 \times \mathrm{ev}_4)^*([D]) &= \frac{1}{960}(2h-2)^2[q^{h-1}]\frac{1}{\Delta} \\ &= [q^{h-1}]\frac{1}{240}\theta^2\left(\frac{1}{\Delta}\right) \\ &= [q^{h-1}]\frac{\frac{11}{5}C_2^2 + C_4}{\Delta}. \end{aligned}$$

For the second term of (5.4), observe that $\overline{M}_{0,3}(X,0) = X$ and $(\mathrm{ev}_3 \times \mathrm{ev}_4)^*([D]) = 24[p]$, so this integral reduces to

$$\frac{1}{10} \int_{[\overline{M}_{1,1}(X,\beta)]^{\mathrm{red}}} \mathrm{ev}^*([p]),$$

which can be evaluated by the result of Bryan and Leung to be

$$\frac{1}{10}[q^{h-1}]\frac{\theta C_2}{\Delta} = [q^{h-1}]\frac{-\frac{1}{5}C_2^2 + C_4}{\Delta}.$$

Adding the two terms of (5.4) gives that

$$\langle \lambda_2 \rangle^{K3} = \frac{2C_2^2 + 2C_4}{\Delta},$$

as claimed by the KKV conjecture.

The genus 3 case is significantly more complicated. To start with, the tautological cohomology space containing λ_3 on $\overline{M}_3$ has rank 10. The details of this were worked out by Faber in [6]; 9 of the 10 generators can be chosen to be Q-classes corresponding to boundary strata $(a), (b), \ldots, (i)$ depicted in Figure 6 of [6]. For the last generator, we let $[(j)]_Q$ be the Q-class corresponding to a genus 0 curve with 1 node intersecting a smooth genus 2 curve at a point, with a cotangent line above the intersection point. Then we can write (see [7])

$$\lambda_3 = \frac{1}{504}\left(\frac{1}{2}[(a)]_Q + [(b)]_Q + [(c)]_Q + \frac{3}{10}[(d)]_Q - \frac{2}{5}[(f)]_Q + 2[(g)]_Q + 2[(j)]_Q\right). \tag{5.5}$$

Through arguments similar to those used in the remark after (5.4), we can show that the integrals of all of these classes vanish except for those of $[(a)]_Q, [(d)]_Q, [(e)]_Q$, and $[(j)]_Q$. Since $[(e)]_Q$ does not appear in the decomposition (5.5), this means that we need to calculate three integrals.

First, the class $[(a)]_Q$ can be handled analogously to δ_{00} in the genus 2 case, since (a) is just the locus of genus 0 curves with 3 nodes. We calculate:

$$\begin{aligned}\int_{[\overline{M}_{3,0}(X,\beta)]^{\text{red}}}[(a)]_Q &= \frac{1}{48}(\beta^2)^3[q^{h-1}]\frac{1}{\Delta}\\ &= [q^{h-1}]\frac{1}{6}\theta^3\left(\frac{1}{\Delta}\right)\\ &= [q^{h-1}]\frac{1760C_2^3 + 2400C_2C_4 + 840C_6}{\Delta}.\end{aligned}$$

The class $[(d)]_Q$ is similarly obtained by adding a node to the genus 2 case δ_{01}, so we can compute

$$\begin{aligned}\int_{[\overline{M}_{3,0}(X,\beta)]^{\text{red}}}[(d)]_Q &= \frac{1}{4}\int_{[\overline{M}_{1,3}(X,\beta)]^{\text{red}}\times X}(\text{ev}_1\times\text{ev}_2)^*([D])(\text{ev}_3\times\text{id})^*([D])(\text{id}\times\text{id})^*([D])\\ &= \frac{1}{4}24(2h-2)\langle\tau_0([p])\rangle_\beta^{K3}\\ &= [q^{h-1}]12\theta\left(\frac{\theta C_2}{\Delta}\right)\\ &= [q^{h-1}]\frac{-480C_2^3 + 1440C_2C_4 + 2520C_6}{\Delta}.\end{aligned}$$

Finally, we calculate the integral of the ψ-class $[(j)]_Q$:

$$\begin{aligned}
\int_{[\overline{M}_{3,0}(X,\beta)]^{\mathrm{red}}} [(j)]_Q &= \frac{1}{2}\int_{[\overline{M}_{2,1}(X,\beta)]^{\mathrm{red}}\times X} \psi_1(\mathrm{ev}_1\times \mathrm{id})^*([D])(\mathrm{id}\times\mathrm{id})^*([D]) \\
&= \frac{1}{2}24\langle\tau_1([p])\rangle_\beta^{K3} \\
&= [q^{h-1}]\frac{12T_1}{\Delta} \\
&= [q^{h-1}]\frac{-32C_2^3+192C_2C_4-84C_6}{\Delta}.
\end{aligned}$$

We now use the decomposition (5.5) and the above three calculations to obtain that

$$\begin{aligned}
\langle\lambda_3\rangle^{K3} &= \frac{1}{504\Delta}\Big(\frac{1}{2}(1760C_2^3+2400C_2C_4+840C_6)+\frac{3}{10}(-480C_2^3+1440C_2C_4+2520C_6) \\
&\quad + 2(-32C_2^3+192C_2C_4-84C_6)\Big) \\
&= \frac{\frac{4}{3}C_2^3+4C_2C_4+2C_6}{\Delta},
\end{aligned}$$

as predicted by the KKV conjecture.

Bibliography

[1] S. Bloch and A. Okounkov, *The character of the infinite wedge representation*, Adv. Math. **149** (2000), no. 1, 1–60.

[2] J. Bryan and N. C. Leung, *The enumerative geometry of $K3$ surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), no. 2, 371–410 (electronic).

[3] J. Bryan and R. Pandharipande, *A Maple program for the Gromov-Witten theory of curves*, http://www.math.ubc.ca/ jbryan/gwall.html.

[4] P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[5] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Invent. Math. **145** (2001), no. 1, 59–103.

[6] C. Faber, *Chow rings of moduli spaces of curves. I. The Chow ring of $\overline{\mathcal{M}}_3$*, Ann. of Math. (2) **132** (1990), no. 2, 331–419.

[7] ———, *Intersection-theoretical computations on $\overline{\mathcal{M}}_g$*, Parameter spaces (Warsaw, 1994), Banach Center Publ., vol. 36, Polish Acad. Sci., Warsaw, 1996, pp. 71–81.

[8] ———, *A non-vanishing result for the tautological ring of $\mathcal{M}_g$*, arXiv:math/9711219v1 [math.AG] (1997).

[9] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), no. 1, 173–199.

[10] E. Getzler and R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle*, Nuclear Phys. B **530** (1998), no. 3, 701–714.

[11] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, vol. 1, American Mathematical Society, Providence, RI, 2003.

[12] S. Katz, A. Klemm, and C. Vafa, *M-theory, topological strings and spinning black holes*, Adv. Theor. Math. Phys. **3** (1999), no. 5, 1445–1537.

[13] D. Maulik and R. Pandharipande, personal communication.

[14] ———, *Gromov-Witten theory and Noether-Lefschetz theory*, arXiv:math/0705.1653v1 [math.AG] (2007).

[15] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[16] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I*, arXiv:math/0101147v2 [math.AG] (2001).

[17] ———, *The equivariant Gromov-Witten theory of* $\mathbf{P}^1$, Ann. of Math. (2) **163** (2006), no. 2, 561–605.

[18] ———, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Ann. of Math. (2) **163** (2006), no. 2, 517–560.

[19] ———, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108.

[20] S. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on* $K3$, Nuclear Phys. B **471** (1996), no. 3, 503–512.