DR CYCLE POLYNOMIALITY AND RELATED RESULTS

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0. Introduction

At the end of the appendix of [1] I claimed that the double ramification cycle formula (proved in that paper) can be used to prove that double ramification cycles are polynomial in the a_i inputs. These rough notes are my attempt to finally write down a proof of that claim (Theorem 2) along with various generalizations and related results stemming from discussions with D. Zagier, especially Zagier's spanning tree formula (Theorem 3). This may eventually turn into a published paper joint with Zagier, as per our original plan, but for now this is just intended as something that can be posted on my website and shared with interested parties, so that the proof is out there in some form, however rough.

1. Statements of results

Let G be a (weakly) connected digraph, with vertex set $V(G)$, directed edge set $E(G)$, and head/tail functions $h, t : E(G) \to V(G)$. Also suppose that for each edge $f \in E(G)$ we have a polynomial $P_f(r) \in \mathbb{Q}[r]$. For any positive integer r such that $P_f(r)$ is an integer, we associate to the edge f the set of r consecutive integers

$$S_f = \{P_f(r), P_f(r) + 1, \ldots, P_f(r) + r - 1\}.$$

We always assume that there exists one positive integer r (and hence infinitely many) such that $P_f(r)$ is an integer for all $f \in E(G)$. We call this data $(G, \{P_f\})$ a *graph with edge polynomials*.

Remark. In the case of the original DR formula, the edge polynomials $P_f(r)$ are just 0 and the sets S_f are $\{0, 1, \ldots, r - 1\}$. We set things up in slightly greater generality because some applications of DR polynomiality use variants of the formula (e.g. k-twisted DR or DR with target varieties) that effectively involve various constant shifts of the set S_f. In addition, Zagier proposed using $S_f = \{\frac{1-r}{2}, \ldots, \frac{r-1}{2}\}$ (for r odd) as a more combinatorially natural construction.

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Given a graph with edge polynomials $(G, \{P_f\})$, a positive integer r such that $P_f(r)$ is an integer for all $f \in E(G)$, and integers $A = (a_v)_{v \in V(G)}$ with sum zero, we define a formal power series

$$\mathrm{DR}^r_{G,\{P_f\}}(A) \in \mathbb{Q}[[\{x_f\}_{f \in E(G)}]]$$

by the finite average

$$\mathrm{DR}^r_{G,\{P_f\}}(A) = \frac{1}{r^{h^1(G)}} \sum_{\substack{w_f \in S_f \\ (\partial w)_v \equiv a_v \pmod r}} \prod_{f \in E(G)} \exp(w_f x_f).$$

Here ∂ is the usual boundary map on 1-chains in the 1-complex G, i.e.

$$(\partial w)_v = \sum_{h(f)=v} w_f - \sum_{t(f)=v} w_f.$$

The most basic polynomiality result is that this averaging construction $\mathrm{DR}^r_{G,\{P_f\}}(A)$ is quasi-polynomial in r:

Theorem 1. *Let $(G, \{P_f\})$ be a graph with edge polynomials, let A be a fixed sum zero vector $A = (a_v)_{v \in V(G)}$, and let M be any monomial in the x_f variables. Then for all sufficiently large r, the coefficient of M in $\mathrm{DR}^r_{G,\{P_f\}}(A)$ is a quasi-polynomial in r with period dividing the LCM of all denominators in the coefficients of the $\{P_f\}$. Moreover, the constant term of this quasi-polynomial is well-defined (i.e. does not depend on the congruence class of r).*

This was proved (in slightly lower generality) as the main result of the appendix of [1]. We will reprove it here in a different way.

Given this theorem, define $\mathrm{DR}_{G,\{P_f\}}(A)$ to be the formal power series formed from $\mathrm{DR}^r_{G,\{P_f\}}(A)$ by taking the constant term of the r-quasi-polynomial for each coefficient in the x_f variables. Then we can state the main polynomiality result of these notes:

Theorem 2. *Let $(G, \{P_f\})$ be a graph with edge polynomials, and let M be any monomial in the x_f variables. Then the coefficient of M in $\mathrm{DR}_{G,\{P_f\}}(A)$ is a polynomial in the integer variables a_v.*

(This implies the polynomiality claim at the end of the appendix of [1].)

This polynomiality is actually a consequence of the following theorem:

Theorem 3. *Let $(G, \{P_f\})$ be a graph with edge polynomials. Then*

$$\mathrm{DR}_{G,\{P_f\}}(A) = \sum_{T \text{ spanning tree in } G} \prod_{f \in E(T)} \exp(a_{f,T} x_f) \prod_{f \notin E(T)} \frac{x_f \exp(P_f(0) x_{f,T})}{\exp(x_{f,T}) - 1},$$

where $a_{f,T}$ is the sum of the a_v for v in the connected component of $T \setminus f$ containing $h(f)$, and $x_{f,T}$ is the signed sum of the x_f around the unique cycle in $T \cup f$, oriented in the direction of f.

(This formula was first discovered and proved by D. Zagier. Note that for the basic double ramification cycle formula of [1], $P_f = 0$ so the $\exp(P_f(0)x_{f,T})$ factor is just 1.)

(I plan to add a couple more results about the polynomials $\mathrm{DR}_{G,\{P_f\}}(A)$ to these notes at some point, but for now I'll stop with this formula of Zagier.)

2. Miscellaneous lemmas

In this section we collect two technical lemmas that are not directly related to the DR sums but will be essential parts of our proofs.

First, a simple construction that yields a quasi-polynomial.

Lemma 4. *Let $P(r) \in \mathbb{Q}[r]$ be a polynomial. Let $m > 0$ be an integer such that $mP(r) \in \mathbb{Z}[r]$. For each integer $r \geq 0$, let $0 \leq [P(r)]_r < r$ denote the unique rational number in that interval differing from $P(r)$ by an integer multiple of r. Then for all sufficiently large r, $[P(r)]_r$ is a quasi-polynomial in r with period dividing m whose constant term is $P(0)$ in all congruence classes (mod m).*

Proof. Assume for simplicity that $P(0) \geq 0$ (the case of negative $P(0)$ is similar). Assume $r > mP(0)$. Then

$$P(r) = P(0) + \frac{r}{m} \cdot \frac{m(P(r) - P(0))}{r},$$

so

$$[P(r)]_r = P(0) + \frac{ir}{m}$$

whenever the integer-coefficient polynomial $\frac{m(P(r)-P(0))}{r}$ is congruent to some i (mod m) with $0 \leq i < m$. But the value of this polynomial mod m only depends on the value of r mod m, so we are done. □

Second, an algebraic identity.

Lemma 5. *Let $0 \leq c < k$ be integers. Let $z_1, \ldots, z_k$ be pairwise distinct elements of some field. Then*

$$1 = \sum_{i=1}^{k} z_i^c \prod_{j \neq i} \frac{1 - z_j}{z_i - z_j}.$$

Proof. It suffices to prove this identity over $\mathbb{C}$. If some $z_i = 1$, all the terms on the right except for the ith term vanish and the identity is trivial. Now assume $z_i \neq 1$ for all i and consider the rational function

$$R(z) = \frac{z^c}{1-z} \prod_{j=1}^{k} \frac{1-z_j}{z-z_j}.$$

The meromorphic differential $R(z)dz$ (on the Riemann sphere) has residue at $z = 1$ equal to -1 and has residue at $z = z_i$ equal to the ith term of the right side of the identity. Since $0 \leq c < k$, these are the only poles and the Residue Theorem gives the desired identity. □

3. Proofs of results

First, the plan: after a simple lemma about reversing edge orientations, we will prove a key recurrence (Proposition 7) for $\mathrm{DR}^r_{G,\{P_f\}}(A)$. This recurrence will give us the r-quasi-polynomiality of Theorem 1, and then we can take the r-constant term of the recurrence to get a slightly simpler recurrence (Corollary 8) for $\mathrm{DR}_{G,\{P_f\}}(A)$. We will then prove Theorem 3 by checking that Zagier's formula also satisfies this recurrence. Finally, Theorem 2 will follow immediately from inspection of Theorem 3.

We start by observing that we can reverse the direction of an edge in the digraph G fairly freely.

Lemma 6. *Let $(G, \{P_f\})$ be a graph with edge polynomials. Let $g \in E(G)$ be an edge. Let H be the digraph formed by replacing edge g in G with a new edge g' in the opposite direction, and let $P_{g'}(r) = 1 - r - P_g(r)$. Then $\mathrm{DR}^r_{H,\{P_f\}}(A) = \mathrm{DR}^r_{G,\{P_f\}}(A)$ after the change of variables $x_{g'} = -x_g$. (Assuming Lemma 1, we also have $\mathrm{DR}_{H,\{P_f\}}(A) = \mathrm{DR}_{G,\{P_f\}}(A)$ after the same change of variables.)*

Proof. This follows immediately from the definition of $\mathrm{DR}^r_{G,\{P_f\}}(A)$ because $S_{g'}(r) = \{-x \mid x \in S_g(r)\}$. □

It is also straightforward to check that the tree sum formula in Theorem 3 is consistent with this edge-reversal property.

Now we give the recurrence, stated for convenience in the case where a given vertex v has no outgoing non-loop edges; using the preceding lemma to reverse edges, we can always make a vertex of this form.

Proposition 7. *Let $(G, \{P_f\})$ be a graph with edge polynomials. Let $v \in V(G)$ be a vertex with no outgoing non-loop edges and $k > 0$*

incoming non-loop edges $f_1, \dots, f_k$. Let $x_i = x_{f_i}$ be the edge variables corresponding to these edges. Then

$$\mathrm{DR}^r_{G,\{P_f\}}(A) = \sum_{i=1}^k \exp(\overline{a}_v x_i) \left(\prod_{j\neq i} \frac{1-e^{-rx_j}}{e^{-rx_i}-e^{-rx_j}} \right) \mathrm{DR}^r_{G_i,\{P_f\}}(A_i)|_{x'_{f_j}=x_j-x_i}.$$

Here $\overline{a}_v$ is the smallest integer in $S_{f_1} + \cdots + S_{f_k}$ congruent to a_v (mod r), G_i is the graph formed by contracting the edge f_i in G (leaving the $\{P_f\}$ data unchanged for the other edges), and A_i is the sum zero vertex weight assignment on G_i given by taking A and adding the weights on the vertices that are identified by the contraction. The power series $\mathrm{DR}^r_{G_i,\{P_f\}}(A_i)$ is interpreted as using variables x'_f for $f \in E(G_i)$, and we change variables to the original variable set by letting $x'_{f_j} = x_j - x_i$, and $x'_f = x_f$ for all other edges f.

Proof. The set of weightings (w_f) summed over in $\mathrm{DR}^r_{G,\{P_f\}}(A)$ naturally bijects (by restriction) to the sets of weightings for each of the contracted versions $\mathrm{DR}^r_{G_i,\{P_f\}}(A_i)$. Thus it is sufficient to check the given identity if we fix a choice of (w_f) for G and use just that single term from each of the DR sums. In other words, the left side of this subidentity to be proved is simply

$$\frac{1}{r^{h^1(G)}} \prod_{f\in E(G)} \exp(w_f x_f).$$

Dividing through by this left side and cancelling various factors, we see that it is sufficient to check that

$$1 = \sum_{i=1}^k \exp\left(\left(\overline{a}_v - \sum_{j=1}^k w_{f_j} \right) \cdot x_i \right) \prod_{j\neq i} \frac{1-e^{-rx_j}}{e^{-rx_i}-e^{-rx_j}}.$$

But $\sum_{j=1}^k w_{f_j}$ belongs to $S_{f_1} + \cdots + S_{f_k}$ and is congruent to a_v (mod r), so

$$\sum_{j=1}^k w_{f_j} - \overline{a}_v = cr$$

for some integer $0 \le c < k$. Thus this identity is just Lemma 5 with $z_i = e^{-rx_i}$, and we are done. □

Using this recurrence, it is easy to see the basic r-quasi-polynomiality of Theorem 1.

Proof of Theorem 1. We first check the result for graphs G with a single vertex (and some number of loops). In this case it is easy to see that

$\mathrm{DR}^r_{G,\{P_f\}}(A)$ factors as a product over the loops, and it suffices to check that each coefficient of the power series

$$\frac{1}{r}\sum_{w=P_f(r)}^{P_f(r)+r-1} e^{wx}$$

is a polynomial in r (for r sufficiently large). But the coefficient of x^n is then just

$$\frac{Q(P_f(r)+r)-Q(P_f(r))}{r}$$

for some polynomial Q of degree $n+1$, and this is clearly a polynomial in r.

Now we induct on the number of vertices in G. Flipping edge directions (as in Lemma 6) clearly doesn't affect r-quasi-polynomiality, so we can assume that G has a vertex v of the form required for Proposition 7. Then the graphs G_i appearing in Proposition 7 have fewer edges than G so are handled by induction, and everything in the recurrence other than the number $\overline{a}_v$ is clearly polynomial in r. Let $P(r) = \sum_i P_{f_i}(r)$, and note that

$$\overline{a}_v = P(r) + [a_v - P(r)]_r$$

where $[x]_r$ is the unique integer in $\{0, \ldots, r-1\}$ congruent to x (mod r). Lemma 4 then gives the desired statement. □

Now that Theorem 1 has been proven, it makes sense to take the constant term in r of both sides of the recurrence Proposition 7. Noting that the quasi-polynomial $\overline{a}_v$ has constant term a_v (by Lemma 4), this immediately gives the following corollary.

Corollary 8. *Let $(G, \{P_f\})$ be a graph with edge polynomials. Let $v \in V(G)$ be a vertex with no outgoing non-loop edges and $k > 0$ incoming non-loop edges $f_1, \ldots, f_k$. Let $x_i = x_{f_i}$ be the edge variables corresponding to these edges. Then*

$$\mathrm{DR}_{G,\{P_f\}}(A) = \sum_{i=1}^{k} \exp(a_v x_i)\left(\prod_{j\neq i}\frac{x_j}{x_j - x_i}\right)\mathrm{DR}_{G_i,\{P_f\}}(A_i)|_{x'_{f_j}=x_j-x_i}.$$

Here G_i is the graph formed by contracting the edge f_i in G (leaving the $\{P_f\}$ data unchanged for the other edges), and A_i is the sum zero vertex weight assignment on G_i given by taking A and adding the weights on the vertices that are identified by the contraction. The power series $\mathrm{DR}_{G_i,\{P_f\}}(A_i)$ is interpreted as using variables x'_f for $f \in E(G_i)$, and we change variables to the original variable set by letting $x'_{f_j} = x_j - x_i$, and $x'_f = x_f$ for all other edges f.

We now prove Theorem 3 (Zagier's formula).

Proof of Theorem 3. We first check this formula for graphs G with a single vertex (and some number of loops). This is a straightforward computation. (I'll probably add the details here later though.)

We now induct on the number of vertices in G. Theorem 3 is compatible with flipping edge directions (as in Lemma 6), so we can assume that G has a vertex v of the form required for Corollary 8. It remains to check that Zagier's formula is compatible with this recurrence.

Note that the spanning trees of the contraction G_i are in bijection with the spanning trees of G that include the edge f_i. Fix a spanning tree T of G; we will check the desired identity after restriction to just those terms using T (or a contraction T_i of T). Let I be the set of i such that T includes the edge f_i. Then the identity to be checked is

$$\prod_{f\in E(T)} \exp(a_{f,T}x_f) \prod_{f\notin E(T)} \frac{x_f \exp(P_f(0)x_{f,T})}{\exp(x_{f,T})-1} =$$

$$\sum_{i\in I} \exp(a_v x_i) \left(\prod_{j\neq i} \frac{x_j}{x_j - x_i}\right) \left[\prod_{f\in E(T)\setminus\{f_i\}} \exp(a_{f,T}x'_f) \prod_{f\notin E(T)} \frac{x'_f \exp(P_f(0)x'_{f,T_i})}{\exp(x'_{f,T_i})-1}\right]_{x'_{f_j}=x_j-x_i}.$$

Observe that under the $x'_{f_j} = x_j - x_i$ change of variables, we have $x'_{f,T_i} = x_{f,T}$. This means that many factors cancel on the two sides. Cancelling them yields the identity Lemma 5 again, this time with $k = |I|$, $c = 0$, and the z_i as the x_i variables (reindexed). □

This formula has only polynomial dependence on the a_v, so Theorem 2 then follows by inspection.

References

1. F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publications mathématiques de l'IHÉS **125** (2017), no. 1, 221–266.