PROBLEM SET 10 (DUE ON TUESDAY, DEC 11)

(All Exercises are references to the November 18, 2017 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 11.2.K (most surfaces of degree $d > 3$ have no lines - Vakil gives a detailed outline of how to do this, and the argument is similar to the one used in the proof of Bertini's theorem, but here is an additional note: if you are unfamiliar with the Grassmannian $\mathbb{G}(1,3)$, you can replace it in this proof with a single affine chart $\mathbb{A}^4$, where the closed point $(x_1, x_2, x_3, x_4) \in \mathbb{A}^4$ corresponds to the line between $[1 : 0 : x_1 : x_2]$ and $[0 : 1 : x_3 : x_4]$ in $\mathbb{P}^3$. You will conclude that "most" degree d surfaces have no lines of this form, and then you can finish by noting that the set of lines in $\mathbb{P}^3$ can be covered by finitely many charts of this type.)

Problem 2. Exercise 11.4.C (useful criterion for irreducibility - you will want to use properness to conclude that X is a variety and that some irreducible component of X surjects onto Y, and then use Prop 11.4.1 to show that this is the only irreducible component)

Problem 3. The *tangent cone* at a point p of a scheme is defined as $\operatorname{Spec} \bigoplus_{i \geq 0} \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$, where the direct sum is given a ring structure in the natural way. Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2)$ (two transverse lines) and $Y = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ (a nodal cubic curve). Show that X and Y have isomorphic tangent cones at the origin. (This is one way of making sense of the statement that these two curve singularities are the "same type.")

Problem 4. Do Exercise 12.3.N (assuming Exercise 12.3.M). Then show that the tangent cone of $\operatorname{Spec} \mathbb{Z}[5i]$ at the point $[(5, 5i)]$ is isomorphic to the tangent cone of $\operatorname{Spec} \mathbb{F}_5[x, y]/(xy)$ at the origin (the point $[(x, y)]$). (In other words, the singularity at $[(5, 5i)] \in \operatorname{Spec} \mathbb{Z}[5i]$ can also be thought of as a simple node.)

Problem 5. Let $X = \operatorname{Proj} \mathbb{C}[x, y, z]/(y^2 z - x^3)$, a cubic curve in $\mathbb{P}^2_{\mathbb{C}}$. Let $X^\vee$ be the dual curve (i.e. the closure of the locus of lines in $\mathbb{P}^2_{\mathbb{C}}$ tangent to X at some nonsingular point of X). Show that X and $X^\vee$ are isomorphic.