PROBLEM SET 8 (DUE ON NOV 9)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Chapter 9, Exercise 2 (Gauss's lemma for Dedekind domains)

Problem 2. Chapter 9, Exercise 7 (every ideal in a Dedekind domain can be generated by two elements)

Problem 3. Chapter 9, Exercise 9 (another version of the Chinese Remainder Theorem)

Problem 4. Suppose that A is a Noetherian domain such that the intersection of any two principal ideals is another principal ideal (for example, a unique factorization domain will satisfy this condition). Show that every invertible fractional ideal in A is principal.

Problem 5. Let $A = \mathbb{R}[x, y]/(x^2+y^2-1)$. Show that the ideal class group of A is nontrivial. (In other words, find a non-principal invertible fractional ideal.)

Problem 6. Let A be a Dedekind domain with ideal class group H. Assume that H is finite and let $p > 0$ be a prime number not dividing the order of H. Suppose that $a, b, c \in A$ satisfy

$$a^p = bc \qquad \text{and} \qquad (b, c) = (1) \text{ is the unit ideal.}$$

Show that there exists a unit $\epsilon \in A^*$ and an element $x \in A$ such that

$$b = \epsilon x^p.$$

(Hint: try to use unique factorization of ideals.)