PROBLEM SET 5 (DUE ON TUESDAY, MARCH 16)

(All Exercises are references to the November 18, 2017 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 18.6.K (Bezout's Theorem - if you want, you can just consider the case where X is reduced)

Problem 2. Suppose Z_1, Z_2 are closed subschemes of a projective k-scheme X. Show that

$$\chi(Z_1 \cup Z_2, \mathcal{O}_{Z_1 \cup Z_2}) = \chi(Z_1, \mathcal{O}_{Z_1}) + \chi(Z_2, \mathcal{O}_{Z_2}) - \chi(Z_1 \cap Z_2, \mathcal{O}_{Z_1 \cap Z_2}),$$

where $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are the scheme-theoretic union and intersection of closed subschemes. (In fact, Hilbert polynomials computed with respect to an ample line bundle on the ambient scheme X will satisfy the same identity - compare with Exercise 18.6.P.)

Problem 3. Let $X \subset \mathbb{P}^3_k$ be the union of three distinct lines that all pass through a single point. Is this enough information to determine the arithmetic genus of X?

Problem 4. Exercise 18.6.R (arithmetic genus of a complete intersection of surfaces in $\mathbb{P}^3$ - in this context complete intersection just means that the surfaces being intersected do not share an irreducible component)

Problem 5. Let m, n be positive integers and let $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m,n))$ be a bihomogeneous polynomial of degree (m,n). Compute the arithmetic genus of the curve $X = V(f)$. Your answer for $(m,n) = (d,d)$ should agree with your answer to the previous problem when $(m,n) = (2,d)$ - why is this? (Hint: use the ideal sheaf sequence for X inside $\mathbb{P}^1 \times \mathbb{P}^1$. Also, the notation $\mathcal{O}(m,n)$ is defined in section 16.4.8.)

Problem 6. Show that for $n \geq 3$, the intersection of any two hypersurfaces in $\mathbb{P}^n_k$ is connected. (Hint: if X is the intersection, compute $h^0(X, \mathcal{O}_X) = 1$ using two long exact sequences of cohomology groups).