Today: birational invariants
- start on unramified morphisms

Recall: Two schemes X, Y are birational if they have isomorphic open dense subschemes.

Q: Which smooth proj. varieties are birational?

dim 1: birational $\Longrightarrow$ isomorphic
we've seen one interesting num'l invariant:
$$\text{genus} = h^1(C, \mathcal{O}_C) \underset{R\text{-}R}{=} h^0(C, \omega_C).$$

dim 2: birational $\not\Longrightarrow$ isomorphic, e.g. $\mathbb{P}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.
Can we find numerical birational invariants?
i.e. we want functions $\varphi : \{\text{smooth proj surfaces}\} \to \mathbb{Z}$
s.t. X birational to $Y \Rightarrow \varphi(X) = \varphi(Y)$.

Tempting to consider $h^i(X, \mathcal{O}_X)$ for $i = 1, 2, \ldots, \dim X$.
Not obvious that $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y)$ for X, Y birational.
(but true, at least if k is perfect).

<u>Prop</u>: $h^0(X, \Omega_{X/k})$ is a birational invariant of smooth proj. varieties$/k$.

<u>Pf</u>: Let $\pi : X \dashrightarrow Y$ be a birational equiv.
Rational maps of smooth proj. varieties are defined outside of a codim 2 subset (why? same idea as the Curve-to-Projective extension theorem.)

$\mathcal{O}_{X, \eta_D}$ is a DVR for η_D a point of codim 1.

So we can find a morphism
$\pi' : U \to Y$, agreeing with π, where $X \setminus U$ is codim ≥ 2 in X.

Then the map $(\pi')^* \Omega_{Y/k} \to \Omega_{U/k}$ (used in our exact sequences) gives a map $H^0(Y, \Omega_{Y/k}) \to H^0(U, \Omega_{U/k}) \cong H^0(X, \Omega_{X/k})$

But since $\Omega_{X/k}$ is a vector bundle, any section of $\Omega_{X/k}$ on U must extend to all of X by Algebraic Hartog's Lemma.

So we have a natural map

$$\pi^*: H^0(Y, \Omega_{Y/k}) \to H^0(X, \Omega_{X/k}).$$

Since π has an inverse as a dominant rational map and π^* was natural (commutes with composition), π^* is an isomorphism. $\boxed{///}$

Prop: The previous prop. holds with $\Omega_{X/k}$ replaced by any other natural vector bundle on X. (just need natural maps $\pi^*\Omega_{Y/k} \to \Omega_{X/k}$).

Example: $\underbrace{h^0(X, \Lambda^i \Omega_{X/k})}_{\text{"i-forms"}}$, $\underbrace{h^0(X, \underbrace{K_X}_{\Lambda^{\dim X}\Omega_{X/k}})}_{\text{"geometric genus"}}$, $\underbrace{h^0(X, K_X^{\otimes j})}_{\text{"jth plurigenus"}}$, etc.

are birational invariants of smooth proj. varieties.

Def: The $\underline{\text{Kodaira dimension of } X}$ is the smallest integer k s.t. $\frac{h^0(X, K_X^{\otimes j})}{j^k}$ is bounded as $j \to \infty$.

($k = -\infty$ if $h^0(X, K_X^{\otimes j}) = 0$ for all $j \geq 1$).

Can show that the Kodaira dimension is equal to the max dimension of the images of the rational maps

$$|K_X^{\otimes j}| : X \dashrightarrow \mathbb{P}^N.$$

(Kodaira dim of X is in $\{-\infty, 0, 1, \ldots, \dim X\}$.)

Defs: Let X be a smooth proj variety.

(1) If X has Kodaira dimension equal to $\dim X$, we say X is general type. ("K_X is big")

(2) If $K_X \cong \mathcal{O}_X$ (so Kodaira dimension $= 0$), we say X is Calabi-Yau. ("$K_X = 0$")

(3) If $K_X^\vee$ is ample ($\Longleftrightarrow (K_X^\vee)^{\otimes n}$ is very ample for some $n > 0$), we say X is Fano ("K_X is negative"). (implies Kodaira dim $= -\infty$).

Example: for curves, general type $\Longleftrightarrow g \geq 2$ ($\deg K_X > 0$)

C-Y $\Longleftrightarrow g = 1$ ($\deg K_X = 0$)

Fano $\Longleftrightarrow g = 0$ ($\deg K_X < 0$).

In higher dimensions, not necessarily in one of the three cases.

These three notions are still our intuition for higher-dim varieties.

A little Hodge theory:

Def: Let X be a smooth proj. variety of dim d.
Then the Hodge numbers of X are
$$h^{i,j} = h^{i,j}(X) := h^j(X, \Lambda^i \Omega_{X,k}) \text{ for } 0 \le i,j \le d.$$
We arrange them in the Hodge diamond

$$h^{d,d}$$

$$h^{d,0} \qquad\qquad\qquad h^{0,d}$$

$$h^{1,0} \qquad h^{0,1}$$

$$h^{0,0} = 1 \text{ (if } X \text{ is red}/k = \bar{k}\text{)}$$

birational invariants

Examples:

curve of genus g:
$$\begin{array}{ccc} & 1 = h^1(C,\omega_C) & \\ h^0(C,\omega_C) = g & & g = h^1(C,\mathcal{O}_C) \\ & 1 & \end{array}$$

$\mathbb{P}^2$:
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

deg 3 surface in $\mathbb{P}^3$:
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\mathbb{P}^1 \times \mathbb{P}^1$:
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

deg 4 surface in $\mathbb{P}^3$ "K3 surface":
$$\begin{array}{cccccl} & & 1 & & & \text{— } H^4 \\ & 0 & & 0 & & \text{— } H^3 \\ 1 & & 20 & & 1 & \text{— } H^2 \\ & 0 & & 0 & & \text{— } H^1 \\ & & 1 & & & \text{— } H^0 \end{array}$$

These diamonds are suspiciously symmetric...

Serre duality: $h^{d-i,d-j} = h^{i,j}$ (since $\Lambda^d V \otimes (\Lambda^i V)^\vee \cong \Lambda^{d-i} V$ for $V \cong k^d$)

If $k = \mathbb{C}$: can compare with de Rham cohomology (Dolbeault cohomology) to get $H^m(X;\mathbb{C}) \cong \bigoplus_{i+j=m} H^j(X, \Lambda^i \Omega_{X/k})$.

and $h^{j,i} = h^{i,j}$ from complex conjugation.

$X =$ connected smooth proj surface$/\mathbb{C}$: Hodge diamond is

$$\begin{matrix} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{matrix}$$

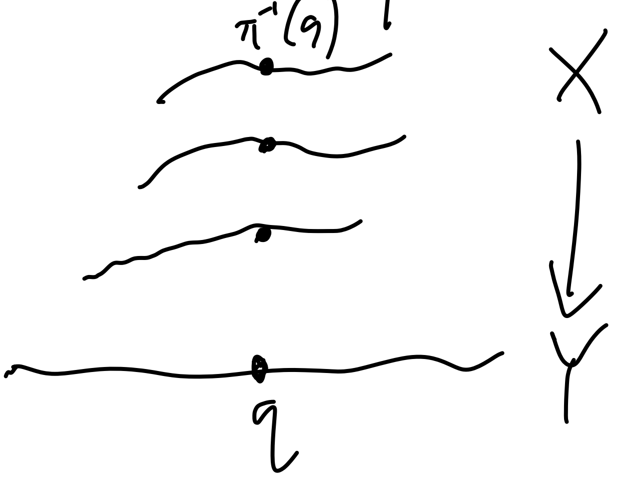
: $p_g =$ "geometric genus" of X
$q =$ "irregularity" of X.

<u>Defs</u>: Suppose $\pi : X \to Y$ is a morphism. The <u>ramification locus</u> of π is the support of $\Omega_{X/Y}$. If $\Omega_{X/Y} = 0$, we say π is <u>formally unramified</u>. If also π is loc. finite type, we say π is <u>unramified</u>.

<u>Examples</u>: open and closed embeddings are unramified.

<u>Lemma</u>: Suppose $\pi : X \to Y$ is loc. finite type. Then π is unramified $\Longleftrightarrow$ for each $q \in Y$, $\pi^{-1}(q)$ is a disjoint union of schemes of the form $\operatorname{Spec} K$, where K/k_q is a finite separable extension.

(Note: if char $k_q = 0$ and π is finite, this condition is equivalent to saying that $\pi^{-1}(q)$ is reduced.)



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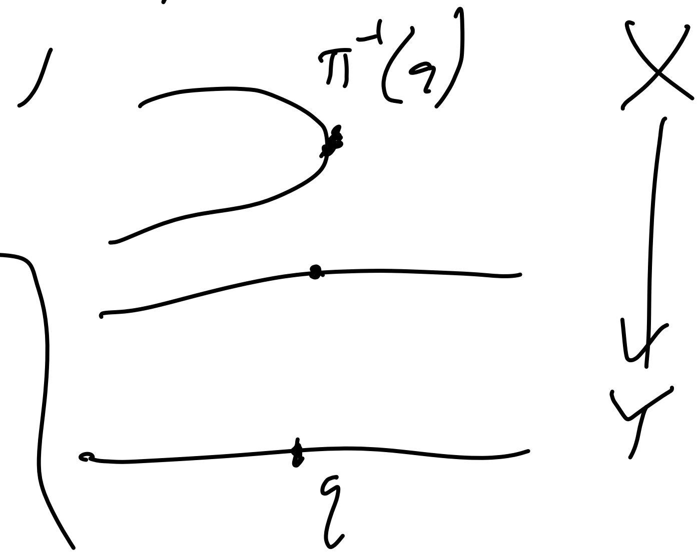

Pf of Lemma:
Classify unramified morphisms

$$X \longrightarrow \operatorname{Spec} k.$$

(trivial cotangent space at every point of X)