Last time: relative cotangent sheaf / sheaf of rel. differentials

$$\Omega_{X/Y} \text{ on } X \quad (\text{for } \pi: X \to Y).$$

Two defs:

affines: $\Omega_{\operatorname{Spec} A/\operatorname{Spec} B} = \Omega_{A/B} = \langle d(\text{generators}) \rangle / \langle d(\text{rels}) \rangle$

$(d(tu) = t\,du + u\,dt)$.

or: $\Omega_{X/Y} = N^\vee_{X/X\times_Y X} = \mathcal{I}/\mathcal{I}^2$ for $\mathcal{I}$ cutting out X in $X \times_Y X$.

Check that the defs agree:

$\operatorname{Spec} A \to \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A$ corresponds to

$$\varphi: A \otimes_B A \to A$$
$$x \otimes y \mapsto xy.$$

$I = \ker \varphi$ is generated by elements $1 \otimes a - a \otimes 1$ for $a \in A$.

Check isomorphisms $\Omega_{A/B} \xleftrightarrow{\sim} I/I^2$

$$da \longmapsto 1 \otimes a - a \otimes 1$$
$$x\,dy \longleftarrow x \otimes y.$$

Examples:

1) $\Omega_{\mathbb{A}^n_k/k} \cong \mathcal{O}_{\mathbb{A}^n_k}^{\oplus n}$ (generated by $dx_1, \dots, dx_n$)

2) $\Omega_{\mathbb{P}^1_k/k} = ?$

$$\mathbb{P}^1_k = \operatorname{Spec} k[t] \underset{t=u^{-1}}{\cup} \operatorname{Spec} k[u].$$

$$\Omega_{\mathbb{P}^1_k/k} = \mathcal{O}_{\mathbb{A}^1_k} \cup \mathcal{O}_{\mathbb{A}^1_k}$$

$$1 \overset{\sim}{\longleftrightarrow} \underbrace{dt \overset{\sim}{\longleftrightarrow} d\left(\frac{1}{u}\right) = -\frac{1}{u^2}\,du}_{} \longleftrightarrow -\frac{1}{u^2}$$

So $\Omega_{\mathbb{P}^1_k/k}$ is a line bundle with a rational section "dt" with

$$\operatorname{div}(dt) = -2[\infty],$$ so

$$\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2).$$

3) (affine plane curve)

$X = \operatorname{Spec} A$, $A = k[x,y]/f$.

$\Omega_{X/k}$ = qcoh sheaf corresp to the A-module

$$(A \cdot dx \oplus A \cdot dy) / \underbrace{df}_{\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} \cdot dy}.$$

Sort of complicated for general f... will be a line bundle usually, not a trivial line bundle though usually.

Some algebra about $\Omega_{A/B}$:

1) $\Omega_{(A/f)/B} \cong \left(\Omega_{A/B} \otimes_A (A/f)\right) / df$

$\left[\begin{tikzcd} B \arrow[r] \arrow[rr, bend left] & A \arrow[r] & A/f \end{tikzcd}\right]$ $\cong \Omega_{A/B} / \left(f \cdot \Omega_{A/B} + A \cdot df\right).$

2) $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$

3) $\Omega_{A \otimes_B C/C} \cong \Omega_{A/B} \otimes_B C.$

Cor (of 2): The stalk $\Omega_{X/k,p} \cong \Omega_{\mathcal{O}_{X,p}/k}.$

Thm: Suppose C is a finite type curve/k and $p \in C$ is a k-valued regular point. Then $\Omega_{C/k,p} \cong \mathcal{O}_{C,p}$. Moreover, if $t \in \mathcal{O}_{C,p}$ generates m_p, then $\underset{\Omega_{\mathcal{O}_{C,p}/k}}{\underset{\cap}{dt}}$ generates $\Omega_{C/k,p}$.

"behaves like $\mathbb{A}^1_k$ near p"

Pf: Let $A = \mathcal{O}_{C,p}$, $m = m_p$, so A is a DVR.

Let t be a uniformizer of A.

Then $M := \Omega_{A/k}$ is a $\underbrace{\text{f.g.}}_{\text{from finite type-ness of } C/k.}$ A-module

Not many options for M since A is a DVR.

Since $A/m = k$ ($k_p = k$), we have

$$A/t^n \cong k[t]/t^n \text{ for all } n \in \mathbb{Z}.$$
$$t \longleftarrow t$$

Then $\Omega_{(A/t^n)/k} \cong \Omega_{A/k} / (t^n \Omega_{A/k} + A \cdot n t^{n-1} dt)$

$$\cong$$

$\Omega_{(k[t]/t^n)/k} \cong (k[t] \cdot dt) / (t^n k[t] \cdot dt + n t^{n-1} k[t] dt)$

So we get $M / t^{n-1} M \cong (A/t^{n-1}) dt$ for all $n > 0$,
$$dt \longleftrightarrow dt$$

So $M \cong A$, gen by dt. $\blacksquare$

Cor: Suppose C is a finite type geom. regular curve/k. Then $\Omega_{C/k}$ is a line bundle.

Pf: Can check being a line bundle after base change to $\bar{k}$. Then $\Omega_{\bar{C}/\bar{k}}$ is a finite type qcoh sheaf with 1-dim fiber at every closed point. $\boxed{\checkmark}$.

With a little more work, can generalize all this from regular points on curves to regular points on finite type k-schemes.

Thm: Suppose X is a finite type k-scheme of pure dim n. Then X is geom. regular $\Longleftrightarrow$ $\Omega_{X/k}$ is a rank n vector bundle.

Def: X is smooth/k (of dim n) if X is a loc. finite type k-scheme and $\Omega_{X/k}$ is a (rank n) vector bundle.

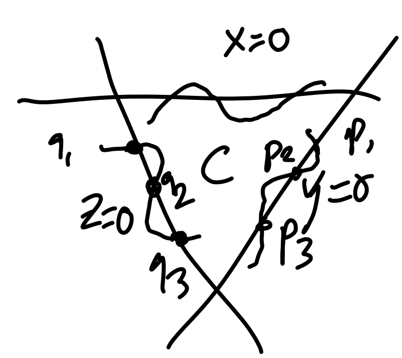
Back to examples:

$C = \mathrm{Proj}\ k[x,y,z]/(x^3+y^3+z^3)$ = smooth genus 1 curve
(char $k \neq 3$).

What is the line bundle $\Omega_{C/k}$? (expect $\mathcal{O}_C$ by R-R etc, since $\Omega_{C/k} = \omega_C$)

Consider the rational section

$$s = d\left(\frac{x}{z}\right) \quad (\text{section on } D(z)).$$



What is $\mathrm{div}(s)$?

1) At the 3 points on C with $z=0$,
$\frac{x}{z}$ has a simple pole, so
$d\left(\frac{x}{z}\right)$ has a double pole.

2) at the 3 points on C with $y=0$,
$\underbrace{\frac{x}{z} - \text{constant}}_{\sim y^3}$ will vanish to order 3; so
$d\left(\frac{x}{z}\right)$ has a double zero.

3) at all other points on C, $\frac{x}{z}$ - constant will be a uniformizer,
so $d\left(\frac{x}{z}\right)$ does not have a zero or pole.

Conclude that

$$\operatorname{div}\left(s = d\left(\tfrac{x}{z}\right)\right) = 2(p_1+p_2+p_3) - 2(q_1+q_2+q_3).$$

But this is equal to $\operatorname{div}\left(\frac{y^2}{z^2}\right)$, so get

$$\Omega_{C/k} \cong \mathcal{O}_C \quad \text{and} \quad H^0(C, \Omega_{C/k}) = k \cdot \frac{z^2}{y^2} \cdot d\left(\frac{x}{z}\right).$$

(Fun check: how does S_3 act on ?)

$$= \frac{x^2}{z^2} \cdot d\left(\frac{y}{x}\right).$$

So direct computations with differentials are possible — but we want some additional techniques.

Two useful exact sequences:

Thm (rel. cotangent exact sequence):
If $X \xrightarrow{\pi} Y \to Z$ are morphisms, then

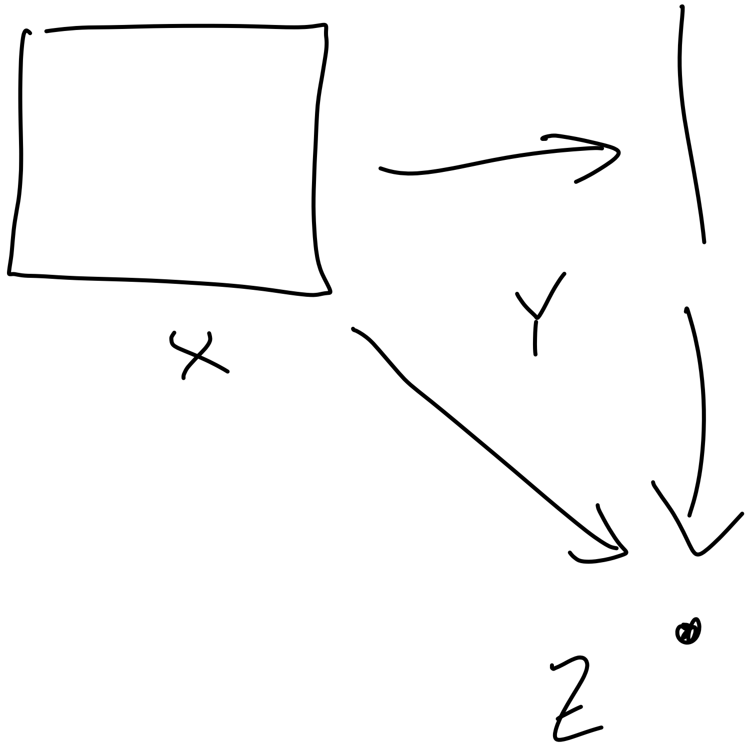
$$\pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact (sheaves on X)

Thm (conormal exact sequence):
If $X \overset{\pi}{\hookrightarrow} Y \to Z$ are morphisms and π is a closed embedding, then

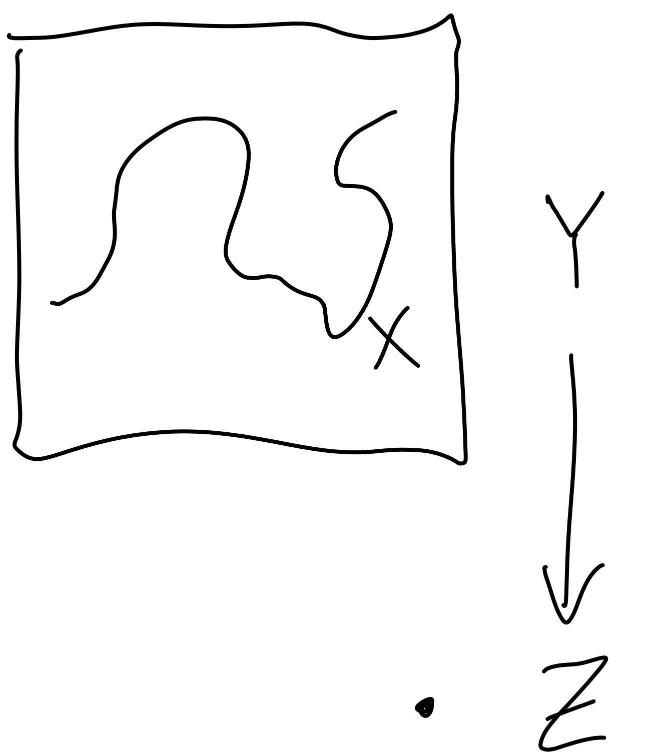
$$\mathcal{N}^\vee_{X/Y} \to \pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

is exact (sheaves on X).



relative cotangent sequence

"vertical $\longrightarrow$ all $\longrightarrow$ horizontal $\longrightarrow$ 0"



"conormal (to X) $\longrightarrow$ all (inside Y, to points of X) $\longrightarrow$ cotangent (to X) $\longrightarrow$ 0"