Next ~2 weeks: curves, now that we have

genus $g = 1 - \chi(C, \mathcal{O}_C)$

and assuming Riemann-Roch.

Today: go over our tools, case $g=0$.

Thm (17.4 in Vakil) Let C be an integral $\underbrace{\text{curve}/k}_{\text{pure 1 dim, finite type}/k}$. Then there is a unique birational equiv (rational map that is an isom of dense opens)

$\pi : C \dashrightarrow C'$, where

C' is a regular proj integral curve/k.

Moreover, if C is regular, π is an open embedding

So for purposes of classifying curves, we usually break it into two problems: $\underbrace{\text{integral regular proj curves}}_{\text{studying using } h^0(C,\mathcal{L}),\ \text{Riemann-Roch, etc}}$, and $\underbrace{\text{curve singularities}}_{\text{we might get to more about this later.}}$

Sketch of pf:

Existence: First pass to an affine open to assume C affine.

Then consider

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\text{closed}"] & \overline{C} \arrow[d, "\text{closed}"] & \widetilde{C} \arrow[l, "\varphi"', "\text{normalization}"] \\ \mathbb{A}^n_k \arrow[r, hook] & \mathbb{P}^n_k & \end{tikzcd}$$

] regular

The normalization φ is finite, hence projective, hence

$$\begin{tikzcd} \widetilde{C} \arrow[r] \arrow[rrr, bend right, "\text{is projective}"'] & \overline{C} \arrow[r, hook] & \mathbb{P}^n_k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

.

<u>Uniqueness</u>: If C_1, C_2 are regular, proj, and $\pi: C_1 \dashrightarrow C_2$ is a birational equiv, then the Curve-to-Projective extension theorem extends this to a morphism $C_1 \longrightarrow C_2$.

Repeat this with π^{-1} to see that it is an isom. $\blacksquare$

Standing assumptions from now on:

C/k is an integral regular proj. curve,

really if $k \neq \bar{k}$ want C to be

<u>geom. integral</u> and <u>geom. regular</u>,

i.e. the base change $C_{\bar{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$

should be integral and regular.

(Recall: $\mathbb{P}^1_{\mathbb{C}}$ is not geom. integral over $\mathbb{R}$ because

$$\mathbb{P}^1_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C} \cong \mathbb{P}^1_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$$

$$\cong \mathbb{P}^1_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} (\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C})$$

$$\cong \mathbb{P}^1_{\mathbb{C}} \sqcup \mathbb{P}^1_{\mathbb{C}}.$$

$h^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = 2$ as an $\mathbb{R}$-scheme

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Our strategy: understand C via morphisms $C \longrightarrow \mathbb{P}^n_k$, i.e. by base-point-free linear systems on C.

Lemma 1: The complete linear system $|\mathcal{L}|$ is base-point-free $\Longleftrightarrow$

$$h^0(C, \mathcal{L}(-p)) < h^0(C, \mathcal{L}) \text{ for all closed } p \in C.$$

$$\mathcal{L}(-p) = \mathcal{L} \otimes \mathcal{O}_C(\underbrace{-p}_{\text{Weil divisor on } C})$$

Pf: $H^0(C, \mathcal{L}(-p)) = \{ s \in H^0(C, \mathcal{L}) \mid \underbrace{\operatorname{div}(s) - p \geq 0}_{s(p)=0} \}$.

So this condition says that there exists a section of $\mathcal{L}$ not vanishing at p. $\blacksquare$.

Lemma 2: Suppose $|\mathcal{L}|$ is base-point-free. Then the morphism $|\mathcal{L}| : C \longrightarrow \mathbb{P}^n_k$ is injective on closed points and injective on tangent vectors at closed points

$$\Longleftrightarrow h^0(C, \mathcal{L}(-p-q)) < h^0(C, \mathcal{L}(-p))$$

for all closed $p, q \in C$ (maybe equal).

Pf:

$p \neq q$: $\exists\, s \in H^0(C, \mathcal{L})$ with $s(p) = 0$ but $s(q) \neq 0$

$\rightsquigarrow$ injective on closed points.

Two reasons why $k=\bar{k}$ can be convenient:

<u>Lemma</u>: If $k=\bar{k}$ then

$$h^0(C,\mathcal{L})-h^0(C,\mathcal{L}(-p)) = 0 \text{ or } 1.$$

<u>Pf</u>: $0 \to \mathcal{L}(-p) \to \mathcal{L} \to \iota_* \iota^* \mathcal{L} \to 0$

$\underset{\text{long exact sequence}}{\rightsquigarrow}$ $0 \to H^0(C,\mathcal{L}(-p)) \to H^0(C,\mathcal{L}) \to H^0(p,\mathcal{L}|_p)$

$$H^0(p,\mathcal{L}|_p) \cong k_p = k. \quad \blacksquare$$

<u>Thm</u>: Suppose $k=\bar{k}$. If $\pi: X \to Y$ is a projective morphism of finite type k-schemes that is injective on closed points and on tangent vectors at closed points, then π is a closed embedding.

<u>Pf</u>: some comm. algebra.

Examples that conditions are necessary:

$\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, $k=\mathbb{R}$

$\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to \operatorname{Spec} k$, or $\operatorname{Spec} \overline{\mathbb{F}}_p[t] \to \operatorname{Spec} \overline{\mathbb{F}}_p[t]$

$$t^p \longleftarrow\!\shortmid\; t$$

"Frobenius"

Riemann-Roch:

Thm: Suppose C has genus g. There exists a line bundle ω_C on C s.t. for any line bundle $\mathcal{L}$ on C,

$$h^0(C,\mathcal{L}) - h^0(C, \omega_C \otimes \mathcal{L}^\vee) = \deg(\mathcal{L}) - g + 1.$$

Pf: Delayed until end of term.

One note: ω_C is canonical.

Lemma: $\deg(\omega_C) = 2g-2$.

Pf.

$$\begin{aligned} & h^0(C,\mathcal{O}_C) - h^0(C,\omega_C) = 0 - g + 1 \\ + \big(& h^0(C,\omega_C) - h^0(C,\mathcal{O}_C) = \deg(\omega_C) - g + 1 \big) \\ \hline & 0 = \deg(\omega_C) - 2g + 2. \ \blacksquare \end{aligned}$$

Cor: If $\deg \mathcal{L} = d > 2g-2$, $h^0(C,\mathcal{L}) = d-g+1$.

Cor: If $\deg \mathcal{L} \geq 2g$, $\mathcal{L}$ is base-point-free.

Cor: If $\deg \mathcal{L} \geq 2g+1$, $\mathcal{L}$ is very ample

even if $k \neq \bar{k}$.

Genus 0:

Thm: If $\boxed{\bar{k}=k}$ and $g(C)=0$, then
$$C \cong \mathbb{P}^1_k.$$

(Note: $\mathbb{P}^1_k$ indeed has genus 0 because
$$1-g = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) - h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1-0.)$$

Pf: Let p be a closed point of C. Then $\mathcal{O}_C(p)$ is a deg 1 line bundle (since $k_p = k$), R-R then gives
$$h^0(\mathcal{O}_C(p)) = 2 \text{ and } \mathcal{O}_C(p) \text{ is very ample.}$$
Then $|p| = |\mathcal{O}_C(p)| : C \hookrightarrow \mathbb{P}^1_k$ is a closed embedding, so an isomorphism. $\square$

Two variants:

Prop: Suppose $\mathcal{L}$ is a degree 1 line bundle on C and $h^0(C,\mathcal{L})=2$. Then
$$C \cong \mathbb{P}^1_k \quad \text{has genus } 0.$$

Pf: Check $\mathcal{L}$ is very ample. $\square$

Prop: Suppose C is a genus 0 curve with a k-valued point (i.e. a closed point p with $k_p = k$). Then
$$C \cong \mathbb{P}^1_k.$$

What if $k \neq \bar{k}$? Then C need not have a k-valued point, e.g.
$$C = \operatorname{Proj} \mathbb{R}[x,y,z]/(x^2+y^2+z^2) \not\cong \mathbb{P}^1_{\mathbb{R}}.$$

Prop: Suppose $g(C)=0$. Then C is isomorphic to a conic in $\mathbb{P}^2_k$.

Pf: There exists a deg 2 line bundle on C:

ω_C has degree $2g-2=-2$, so $\mathcal{L}=\omega_C^\vee$ has degree 2.

Then R-R gives $h^0(C,\mathcal{L})=3$ and $\mathcal{L}$ is very ample, so $|\mathcal{L}|: C \hookrightarrow \mathbb{P}^2_k$ identifies C with a conic (since $\deg(\mathcal{L})=2$). $\blacksquare$.

Note: Can use this to show if char $k \neq 2$,

$$\{\text{genus } 0 \ C/k\}/\text{isom} \xleftrightarrow{\text{bij}} k^*/(k^*)^2.$$

Plan for $g \geq 2$: repeat this trick, but use ω_C instead of $\omega_C^\vee$ to understand C.

<u>Lemma:</u> Suppose $g(C) > 0$. Then ω_C is base-point-free.

<u>Pf:</u> After base changing, can assume $k = \bar{k}$. Then it suffices to check $h^0(C, \omega_C(-p)) = h^0(C, \omega_C) - 1$ for all closed points p.

By Riemann-Roch, this is the same as

$$h^0(C, \mathcal{O}_C(p)) = h^0(C, \mathcal{O}_C).$$

This is true, since $h^0(C, \mathcal{O}_C) = 1$ and

$$h^0(C, \mathcal{O}_C(p)) - h^0(C, \mathcal{O}_C) = 0 \text{ or } 1,$$

so if $h^0(C, \mathcal{O}_C(p)) \neq h^0(C, \mathcal{O}_C)$, then $h^0(C, \mathcal{O}_C(p)) = 2$ and by previous criterion, C has genus 0. $\Rightarrow\!\Leftarrow$ $\boxed{/\!/}$

By this lemma, if $g(C)>0$, we have a canonical morphism

$$|\omega_C| : C \longrightarrow \mathbb{P}^{g-1}$$

$h^0(C, \omega_C) = h^1(C, \mathcal{O}_C) = g.$