Last time: $\chi(X, \mathcal{F})$,

$$h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = \chi(C,\mathcal{L}) = \deg \mathcal{L} - g + 1$$

$$h^1(C,\mathcal{L}) = h^0(C, \underbrace{\omega_C \otimes \mathcal{L}^\vee}_{\text{new line bundle}})$$ (Riemann-Roch, a special case of Serre duality)

Today: The Hilbert polynomial $p_{\mathcal{F}}(n)$.

(return to curve theory next week)

Thm/Defs: Let $\mathcal{F}$ be a finite type qcoh sheaf on a proj k-scheme X with very ample line bundle $\mathcal{O}(1)$ (i.e. a chosen embedding $X \hookrightarrow \mathbb{P}^N_k$).

Then $p_{\mathcal{F}}(m) := \chi(X, \mathcal{F}(m))$ is a polynomial in m, called the Hilbert polynomial of $\mathcal{F}$. The leading term of $p_{\mathcal{F}}(m)$ is $\frac{d_{\mathcal{F}}}{n!} m^n$, where

$d_{\mathcal{F}}$ is a positive integer (unless $\mathcal{F}=0$) &

$n = \dim \operatorname{Supp}(\mathcal{F})$ $[\operatorname{Supp}(\mathcal{F}) = \{p \in X \mid \mathcal{F}_p \neq 0\}]$.

When $\mathcal{F} = \mathcal{O}_X$, we write $p_X(m) := p_{\mathcal{O}_X}(m)$ and call it the Hilbert poly of X; the positive integer $d_X = d_{\mathcal{O}_X}$ is called the degree of X.

(Warning: $p_X(m)$ and d_X still depend on $X \hookrightarrow \mathbb{P}^N$.)

Should think of $p_X(m)$ as a fundamental invariant of closed subschemes of $\mathbb{P}^N_k$.

Examples:

1) $X = \mathbb{P}^n_k$: we previously computed

$$p_X(m) = \chi(\mathbb{P}^n, \mathcal{O}(m)) = \binom{m+n}{n}$$

$$= \frac{1}{n!}(m+n)(m+n-1)\cdots(m+1), \text{ so } d_{\mathbb{P}^n} = 1.$$

2) $X = V(f) \subset \mathbb{P}^n_k$, a hypersurface of deg d:

We have short exact sequences on $\mathbb{P}^n_k$

$$0 \to \mathcal{I}_X = \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \iota_* \mathcal{O}_X \to 0$$

$\boxed{\iota : X \hookrightarrow \mathbb{P}^n_k}$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-d) \to \mathcal{O}_{\mathbb{P}^n}(m) \to (\iota_* \mathcal{O}_X)(m) \to 0$$

$$(\iota_* \mathcal{O}_X)(m) = \iota_*(\mathcal{O}_X(m))$$

So $p_X(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}$

$$= \frac{d}{(n-1)!} m^{n-1} + (\text{lower order}),$$

so indeed $d_{V(f) \hookrightarrow \mathbb{P}^n} = d$.

Pf of Thm (existence of Hilbert polynomial):

First: reduce to case $X = \mathbb{P}^N_k$ (replace $\mathcal{F}$ with $\iota_* \mathcal{F}$ for $\iota : X \hookrightarrow \mathbb{P}^N_k$).

Also can assume k is infinite by base changing if necessary.

Suppose $f \in H^0(\mathbb{P}^N_k, \mathcal{O}(1))$ is a nonzero linear form, so $H := V(f)$ is a hyperplane. Our plan is to take the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1) \xrightarrow{\text{mult by } f} \mathcal{O}_{\mathbb{P}^N} \longrightarrow \iota_* \mathcal{O}_H \longrightarrow 0$$

and tensor with $\mathcal{F}$.

Here $\mathcal{F}$ is not necessarily a vector bundle, so $\otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{F}$ is not necessarily an exact functor,

So we just get an exact sequence

$$\mathcal{F}(-1) \xrightarrow{\text{mult by } f} \mathcal{F} \longrightarrow \underset{\iota_* \iota^* \mathcal{F}}{\underset{\|}{\mathcal{F} \otimes \iota_* \mathcal{O}_H}} \longrightarrow 0$$

Claim: We can choose H (equiv choose f) so that this sequence will still be exact on the left.

(Pf of claim: $\mathcal{F} = \widetilde{M_\bullet}$ for some graded module $M_\bullet$,
use comm. alg. to find a non-zero-divisor for $M_\bullet$

Vakil discusses this in terms of "associated points"
– geometric way to think about zero-divisors of modules
(section 5.5))

Even without the claim, we have

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{F} \to \iota_*(\iota^*\mathcal{F}) \to 0$$

for some $\mathcal{G}$.

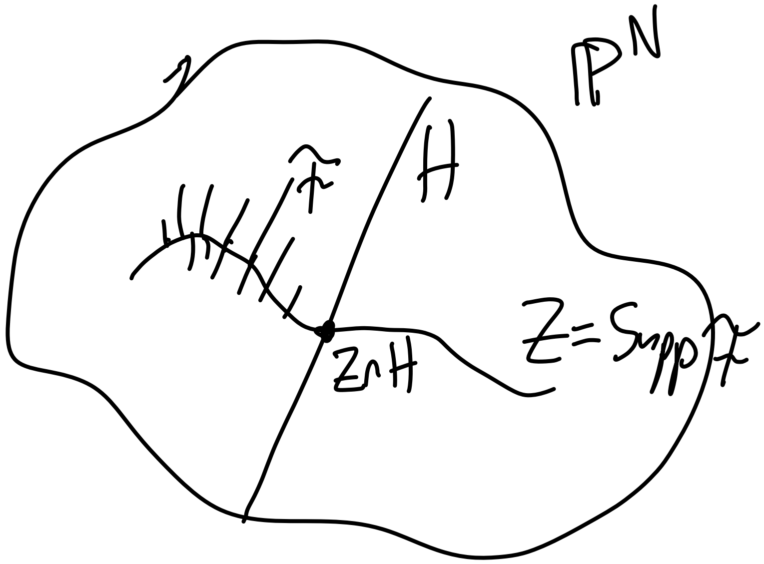
Then $\operatorname{Supp} \mathcal{G}$ and $\operatorname{Supp} \iota_*(\iota^*\mathcal{F})$ are both contained in $H \cap \operatorname{Supp} \mathcal{F}$, since mult by f is invertible away from $H = V(f)$.

Choosing H with $\dim(H \cap \operatorname{Supp}\mathcal{F}) = \dim \operatorname{Supp}\mathcal{F} - 1$ then gives

$$P_{\mathcal{F}}(m) - P_{\mathcal{F}}(m-1) = \underbrace{P_{\iota_*\iota^*\mathcal{F}}(m) - P_{\mathcal{G}}(m)}_{\text{poly of deg} \leq \dim \operatorname{Supp}\mathcal{F} - 1 \text{ by induction on } \dim \operatorname{Supp}\mathcal{F}}$$

$P_{\mathcal{F}}(m)$ is a poly of $\deg \leq \dim \operatorname{Supp}\mathcal{F}$. $\Longleftarrow$ poly of $\deg \leq \dim \operatorname{Supp}\mathcal{F} - 1$ by induction on $\dim \operatorname{Supp}\mathcal{F}$

$\boxed{/\!/}$.



For most H, this works well to compute $P_{\tilde{\mathcal{F}}}(n) - P_{\tilde{\mathcal{F}}}(n-1)$ in terms of simpler geometry $(\iota_H)_* \iota_H^* \tilde{\mathcal{F}}$, supp on $Z \cap H$.

Summary:

$$X \underset{\text{closed}}{\hookrightarrow} \mathbb{P}^N_k \rightsquigarrow p_X(m) \in \mathbb{Q}[m]$$

$$\| $$

$$\frac{d_X m^{\dim X}}{(\dim X)!} + \cdots + \chi(X, \mathcal{O}_X)$$

$$p_X(m) = \chi(X, \mathcal{O}_X(m)) \underset{m \gg 0}{=} h^0(X, \mathcal{O}_X(m))$$

Serre vanishing

We already know: if $X = V(I)$ for some homog. ideal $I \subseteq k[t_0, \ldots, t_N]$

then for $m \gg 0$ we have

$$h^0(X, \mathcal{O}_X(m)) = (k[t_0, \ldots, t_N]/I)_m$$

So

$$\underbrace{p_X(m) = \chi(X, \mathcal{O}_X(m))}_{\substack{\text{polynomial for all } m \in \mathbb{Z} \\ \text{constant term is } \chi(X, \mathcal{O}_X) = 1 - p_a(X)}} \underset{m \gg 0}{=} \underbrace{h^0(X, \mathcal{O}_X(m)) \underset{m \gg 0}{=} \overbrace{(k[t_0, \ldots, t_N]/I)_m}^{\text{very convenient for direct computation}}}_{\substack{\text{polys for } m \gg 0 \\ \text{constant term of poly is mysterious}}}$$

<u>compatibility</u> with other notions of "degree":

1) deg d hypersurfaces are deg d - already done.

2) If C is a regular proj curve and $\mathcal{L}$ is a very ample line bundle on C, then

$\underbrace{d_C}_{\text{embedded using } \mathcal{L}} = \deg \mathcal{L}$ because $\chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C) = \deg \mathcal{L}$

$$\chi(C,\mathcal{L}) - \chi(C,\mathcal{O}_C) = P_C(1) - P_C(0) = d_C$$

because $P_C(m)$ is a linear polynomial.

3) Suppose $\pi : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ is a morphism with $\pi^* \mathcal{O}(1) = \mathcal{O}(d)$. If π is a closed embedding, it is tempting to guess that d is the degree of π (in the Hilbert poly sense). (Sometimes this is given as the def of the degree of a morphism $\pi : \mathbb{P}^n \to \mathbb{P}^N$.)

These notions are <u>not</u> equal.

<u>Q</u>: What is the Hilbert polynomial of the Veronese embedding $v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$?

$$v_d = |\mathcal{O}(d)|$$

$$P_{\mathbb{P}^n \xrightarrow{v_d} \mathbb{P}^N}(m) = \chi(\mathbb{P}^n, (v_d^* \mathcal{O}_{\mathbb{P}^N}(1))^{\otimes m})$$

$$= \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dm)) = P_{\mathbb{P}^n \xrightarrow{id} \mathbb{P}^n}(dm)$$

$$= \binom{dm+n}{n}$$, so degree is actually d^n, not d.

Back to curves:

Our Hilbert poly calculation

$$P_{V(f)}(m) = \binom{m+n}{n} - \binom{m+n-d}{n} \text{ for } f \text{ of degree } d$$

immediately gives us the genus of a deg d plane curve.

Lemma: A degree d plane curve has (arithmetic) genus

$$\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

Pf: $g = 1 - \chi(C, \mathcal{O}_C) = 1 - P_C(0) = 1 - \binom{0+2}{2} + \binom{0+2-d}{2}$

$$= 1 - 1 + \frac{(2-d)(1-d)}{2} = \binom{d-1}{2}. \quad \blacksquare$$

Cor: There exist regular proj. curves/$\mathbb{C}$ of genus

$0, 1, 3, 6, 10, 15, \ldots$

($\mathrm{Proj}\ \mathbb{C}[x,y,z]/(x^d + y^d + z^d)$ is a regular curve).

Cor: There are ∞-ly many nonisomorphic connected projective regular curves/$\mathbb{C}$.

[Moral: cohom helps us distinguish schemes from each other]

Another way of viewing this: now have a concrete way in which deg d plane curves became "more complicated" as d increases.

Q: How can we get curves of arbitrary genus, not just $\binom{d-1}{2}$?

A: (1) next pset: Look inside $\mathbb{P}^3$, not just $\mathbb{P}^2$ (construction of non-planar curves)

(2) take a singular plane curve of deg d and fix the singularities (by taking the normalization).