Last time:

projective scheme $\iota: \operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ $\rightsquigarrow$ line bundle $\mathcal{O}_{\operatorname{Proj} S}(1) = \iota^* \mathcal{O}_{\mathbb{P}^n}(1)$

$= \mathcal{O}_X(X \cap \text{hyperplane})$

Today: ① a bit more about pullbacks of quasicoherent sheaves

② morphisms to $\mathbb{P}^n$ via line bundles

$$\left.\begin{array}{l}\pi: X \to Y \\ \mathcal{F} \text{ qcoh sheaf } Y\end{array}\right] \rightsquigarrow \pi^* \mathcal{F} \text{ qcoh sheaf on } X.$$

Previous description: if $\operatorname{Spec} A \subseteq \pi^{-1}(\operatorname{Spec} B) \subseteq X$,

take $(\pi^* \mathcal{F})(\operatorname{Spec} A) := \mathcal{F}(\operatorname{Spec} B) \otimes_B A$

(difficulty in making this into a rigorous def: show changing $\operatorname{Spec} B$ only affects $\mathcal{F}(\operatorname{Spec} B) \otimes_B A$ up to canon. isos)

Let's discuss a less affine-centric approach to defining π^*.

$\pi: X \to Y$

Idea: $\pi_*: QCoh_X \longrightarrow QCoh_Y$ ← cat. of qcoh sheaves on Y

(assuming π is qcqs, or can use cats of $\mathcal{O}_Y$-modules.)

$\pi^*: QCoh_Y \longrightarrow QCoh_X$

should form an adjoint pair, i.e.

$$\mathrm{Hom}_{QCoh_Y}(\mathcal{F}, \pi_* \mathcal{G}) \cong \mathrm{Hom}_{QCoh_X}(\pi^* \mathcal{F}, \mathcal{G})$$

(Why? analogous to $\mathrm{Hom}_B(M, N) \cong \mathrm{Hom}_A(M \otimes_B A, N)$,

where A is a B-algebra, M is a B-module, N is an A-module)

Approach: find similar adjoint pair in categories of sheaves, then move the result to $\mathcal{O}_Y$-modules.

Def: Given a sheaf $\mathcal{F}$ on Y and a map $\pi: X \to Y$, let $\pi^{-1}\mathcal{F}$ be the sheaf on X given as the sheafification of the presheaf

$$\left((\pi^{-1})^{pre}\mathcal{F}\right)(U) := \left\{ (V, s) \;\middle|\; \begin{array}{l} V \subseteq Y \text{ open} \\ V \supseteq \pi(U) \\ s \in \mathcal{F}(V) \end{array} \right\} \Big/ \underset{\text{restriction}}{\sim}.$$

Claim: $\pi_*: Sets_X \to Sets_Y$ and $\pi^{-1}: Sets_Y \to Sets_X$ are an adjoint pair.

If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then $\pi^{-1}\mathcal{F}$ is just a $(\pi^{-1}\mathcal{O}_Y)$-module, not an $\mathcal{O}_X$-module, but via adjointness, the structure morphism of π, $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ corresponds to a morphism

$$\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X .$$

<u>Def</u>: The <u>pullback</u> of an $\mathcal{O}_Y$-module $\mathcal{F}$ along $\pi : X \to Y$ is

$$\pi^* \mathcal{F} := (\pi^{-1}\mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X .$$

Selected nice properties of pullback (by $\pi: X \to Y$, $\mathcal{F}$ = sheaf on Y)

1) $\pi^*\mathcal{O}_Y \cong \mathcal{O}_X$, and hence if $\mathcal{F}$ is a rank r vector bundle then so is $\pi^*\mathcal{F}$.

2) fiber dimension is preserved, i.e.

$$(\pi^*\mathcal{F})\big|_x \cong \left(\mathcal{F}\big|_{\pi(x)}\right) \otimes_{k_{\pi(x)}} k_x$$

(follows from tensor products composing nicely).

3) distributes over tensor products, i.e.

$$\pi^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \cong (\pi^*\mathcal{F}) \otimes_{\mathcal{O}_X} (\pi^*\mathcal{G}),$$

so get group homomorphism $\pi^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$

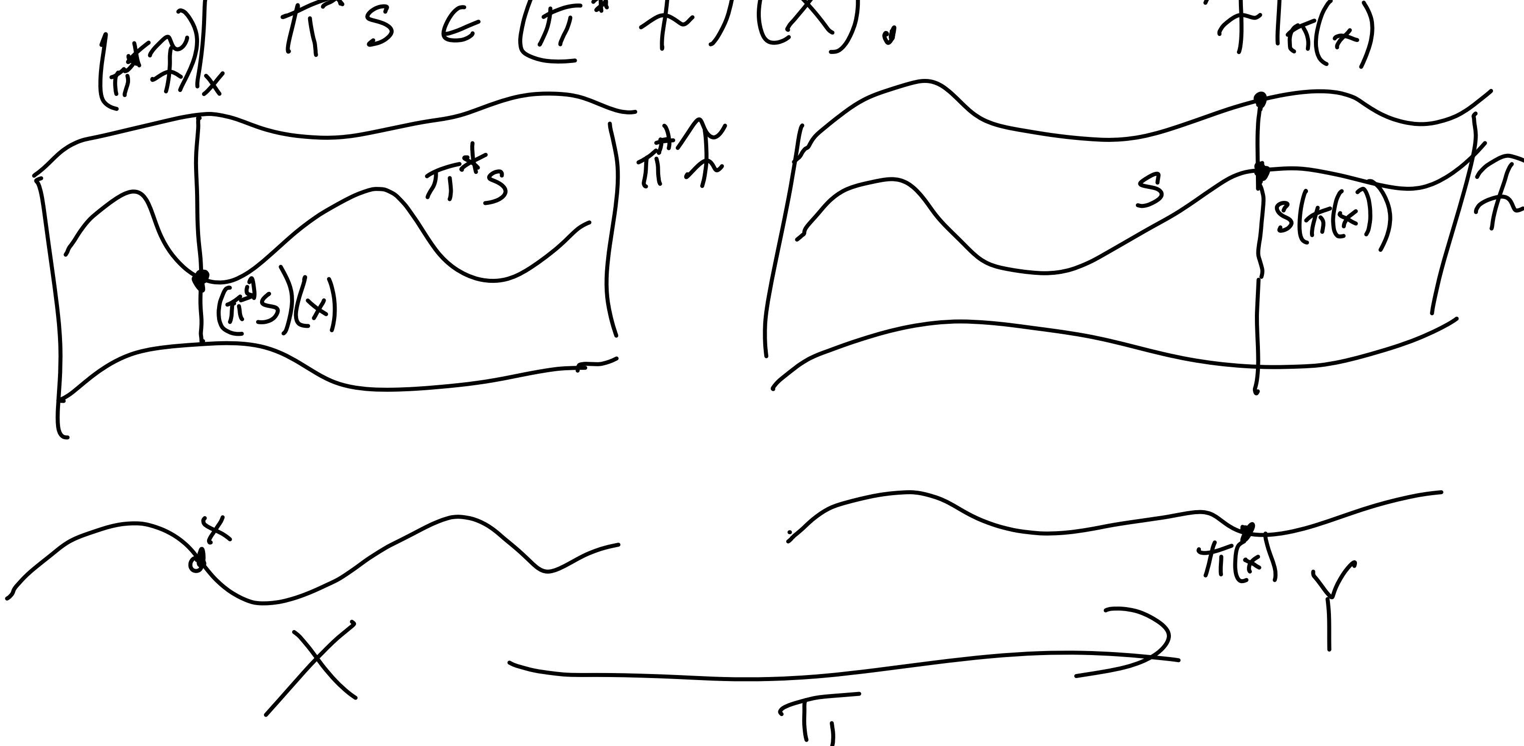
(makes Pic into a functor $\mathrm{Sch} \to \mathrm{Ab}$)

4) functorial on morphisms of sheaves, i.e.

$$\varphi: \mathcal{F} \to \mathcal{G} \rightsquigarrow \pi^*\varphi: \pi^*\mathcal{F} \to \pi^*\mathcal{G}.$$

Since sections $s \in \mathcal{F}(Y)$ correspond to morphisms $\mathcal{O}_Y \to \mathcal{F}$, this means that we can pull back sections and define

$$\pi^* s \in (\pi^* \mathcal{F})(X).$$



Claim: $(\pi^* s)(x) = s(\pi(x)) \otimes 1$ under the isom of fibers from above.

(If $\mathcal{F}$ is a line bundle, this is directly related to the analogous statement about values of pullbacks of functions, since can pass to a trivialization $\mathcal{F} \cong \mathcal{O}$)

In particular, $\pi^* s$ vanishes at x
$\iff$ s vanishes at $\pi(x)$.

Suppose $\mathcal{L}$ is a line bundle on Y and $s \in \mathcal{L}(Y)$, and X, Y are Noetherian + normal.

Then in general it is a little tricky to relate $\operatorname{div}(s) \in \operatorname{Weil} Y$ and $\operatorname{div}(\pi^* s) \in \operatorname{Weil} X$.

Thinking in terms of ideal sheaves (i.e. effective Cartier divisors), there's some discussion of this in Vakil 16.3.9; scheme-theoretically, $V(\pi^* s)$ is not always equal to $\pi^{-1}(V(s))$ (but often is).

But by preceding discussion today, these are at least equal as sets.

Can compute mults for Weil divisors $\operatorname{div}(s)$, $\operatorname{div}(\pi^* s)$ by hand usually.

Morphisms to $\mathbb{P}^n$:

Thm: Let A be a ring. Let X be an A-scheme. Let $n \geq 0$. Then there is a bijection

$$\left\{ \begin{array}{c} A\text{-scheme morphisms} \\ X \to \mathbb{P}^n_A \end{array} \right\} \xleftrightarrow{\sim} \left\{ (\mathcal{L}, s_0, \dots, s_n) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ line bundle on } X \\ s_i \in \mathcal{L}(X) \\ \text{no simultaneous} \\ \text{zeroes} \end{array} \right\}$$

($\Leftrightarrow$ the s_i generate every fiber of $\mathcal{L}$)

$$\pi \longmapsto \left(\pi^* \mathcal{O}_{\mathbb{P}^n}(1), \pi^* x_0, \dots, \pi^* x_n\right)$$

$$\underbrace{[s_0(x) : \dots : s_n(x)]}_{s_i(x) \in V = \mathcal{L}|_x \cong k_x} \longleftarrow\!\shortmid \left(\mathcal{L}, s_0, \dots, s_n\right)$$

Pf: *Idea:* use the open cover of X by the

$$D(s_i) = \{x \in X \mid s_i(x) \neq 0\} = \pi^{-1}(D(x_i))$$

Given $(\mathcal{L}, s_0, \dots, s_n)$, we have trivializations

$$\left.\begin{array}{l} \left(\mathcal{L}|_{D(s_i)}, s_i\right) \cong \left(\mathcal{O}_X|_{D(s_i)}, 1\right) \\ \qquad \cong \left(\pi^* \mathcal{O}(1)|_{D(s_i)}, \pi^* x_i\right) \end{array}\right]$$

can check transition functions (gluing data) are consistent.

$\pi : X \to \mathbb{P}^n$

image not in any coord. hyperplane

$(\mathcal{L}, s_0, \dots, s_n)$

no common zeroes, $s_i \neq 0$

(π)

$(\pi^*\mathcal{O}(1), \pi^* x_i)$

$D_0, \dots, D_n \in (\mathrm{Weil}\, X)_{\geq 0}$

linearly equiv (i.e. equal in $Cl(X)$)

trivial intersection of supports

$\pi^{-1}(V(x_i))$ (really $\mathrm{div}(\pi^* x_i)$)

assume X factorial,